THE RELATIVE HELLER OPERATOR AND RELATIVE COHOMOLOGY FOR THE KLEIN 4-GROUP.

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ABSTRACT. Let G be the Klein Four-group and let $\mathbb{k}$ be an arbitrary field of characteristic 2. A classification of indecomposable $\mathbb{k}G$-modules is known. We calculate the relative cohomology groups $H^i_\chi(G, N)$ for every indecomposable $\mathbb{k}G$-module N, where χ is the set of proper subgroups in G. This extends work of Pamuk and Yalcin to cohomology with non-trivial coefficients. We also show that all cup products in strictly positive degree in $H^*_\chi(G, \mathbb{k})$ are trivial.

1. INTRODUCTION

Let G be a finite group and $\mathbb{k}$ a field of characteristic $p > 0$. If $p \not| |G|$, then every representation of G over $\mathbb{k}$ is projective. Thus, by decomposing the regular module $\mathbb{k}G$ we can obtain all isomorphism classes of $\mathbb{k}G$-modules immediately.

From now on assume $p||G|$. Then the above is no longer true. However, it is well-known that, given a $\mathbb{k}G$-module M, we can find a projective module P_0 and a surjective $\mathbb{k}G$-morphism

$$\pi_0 : P_0 \to M.$$

If we choose P_0 and π_0 so that P_0 has smallest possible dimension, then this pair is unique, and known as the projective cover of M. The kernel of π_0 is denoted $\Omega(M)$. The is known as the Heller shift of M. $\Omega(-)$ can be viewed as an operation on the set of $\mathbb{k}G$-modules which takes indecomposable modules to indecomposable modules.

This construction can be iterated. For each $i > 0$, let $\pi_i : P_i \to \Omega^i(M)$ be the projective cover of $\Omega^i(M)$. By composing these maps with the inclusions $\Omega^i(M) \to P_{i-1}$, we obtain an exact sequence

$$\dots P_i \to P_{i-1} \to \dots \to P_0 \to M \to 0. \tag{1}$$

This is an example of a projective resolution for M. If N is any $\mathbb{k}G$-module, then the above induces a complex

$$0 \to \mathrm{Hom}_{\mathbb{k}G}(P_0, M) \to \dots \to \mathrm{Hom}_{\mathbb{k}G}(P_i, M) \to \dots$$

which is not exact in general. The homology groups of this complex are by definition the groups $\mathrm{Ext}^i_{\mathbb{k}G}(M, N)$. A special case is

$$H^i(G, N) := \mathrm{Ext}^i_{\mathbb{k}G}(\mathbb{k}, N).$$

We call this the *cohomology of G with coefficients in N*.

There is a long and fruitful history of study of the cohomology groups $H^i(G, N)$ in modular representation theory. Further, one may define a pairing

$$\smile: H^i(G, \mathbb{k}) \otimes H^j(G, \mathbb{k}) \to H^{i+j}(G, \mathbb{k})$$

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which gives $H^*(G, \Bbbk)$ the structure of a graded-commutative graded ring. A celebrated theorem of Evens (see [3, Theorem 4.2.1]) states that, for any G, the ring $H^*(G, \Bbbk)$ is finitely generated.

Now let χ be a set of proper subgroups of G. A $\Bbbk G$-module M is said to be *projective relative to* χ if M is a direct summand of $\oplus_{X \in \chi} M \downarrow_X \uparrow^G$. Other equivalent definitions will be given in section 2. It is less well-known, but still true, that every $\Bbbk G$-module has a unique relative projective cover with respect to χ. This is defined to be a $\Bbbk G$-module Q_0 of smallest dimension such that

(1) Q_0 is projective relative to χ;
(2) There is a surjective $\Bbbk G$-morphism $\pi_0 : Q_0 \to M$ which splits on restriction to each $X \in \chi$.

The kernel of Q_0 is denoted $\Omega_\chi(M)$ and called the relative Heller shift of M with respect to χ. We can mimic the construction of (1) to obtain a relative projective resolution of M, that is, an exact sequence

$$\ldots Q_i \to Q_{i-1} \to \ldots \to Q_0 \to M \to 0. \tag{2}$$

of $\Bbbk G$ modules which are projective relative to χ and in which the connecting homomorphisms split over each $X \in \chi$. Given any $\Bbbk G$-module M, the above induces a complex

$$0 \to \mathrm{Hom}_{\Bbbk G}(Q_0, M) \to \ldots \to \mathrm{Hom}_{\Bbbk G}(Q_i, M) \to \ldots$$

which is in general no longer exact. The homology groups of this complex are by definition the relative Ext-groups $\mathrm{Ext}^i_{\Bbbk G, \chi}(M, N)$. The relative cohomology of G with respect to χ with coefficients in N is the special case

$$H^i_\chi(G, N) := \mathrm{Ext}^i_{\Bbbk G, \chi}(\Bbbk, N).$$

Further, one may define a pairing

$$\smile : H^i_\chi(G, \Bbbk) \otimes H^j_\chi(G, \Bbbk) \to H^{i+j}_\chi(G, \Bbbk)$$

which gives $H^*(G, \Bbbk)$ the structure of a graded-commutative graded ring.

Computations of $H^i_\chi(G, N)$ are rare in the literature. It is notable that the ring $H^*_\chi(G, \Bbbk)$ is not finitely generated in general. This was first discovered by Blowers [4], who showed that if G_1 and G_2 are finite groups of order divisible by p, and χ_1, χ_2 are sets of subgroups of G_1, G_2 respectively with order divisible by p, then all products of elements of positive degree in $H^*_\chi(G, \Bbbk)$ are zero, where $G = G_1 \times G_2$ and $\chi = \{G_1 \times X : X \in \chi_2\} \cup \{X \times G_2 : X \in \chi_1\}$. See also [5].

For the rest of this section, let $G = \langle \sigma, \tau \rangle$ denote the Klein four-group, and let $\Bbbk$ be a field of characteristic 2. We set $\chi = \{H_1, H_2, H_3\}$, the set of all proper nontrivial subgroups of G, where $H_1 = \langle \sigma \rangle, H_2 = \langle \tau \rangle, H_3 = \langle \sigma\tau \rangle$.

The cohomology groups $H^i_\chi(G, \Bbbk)$ were computed, by indirect means, by Pamuk and Yalcin [9]. In the present article we recover their result, and also compute $H^i_\chi(G, N)$ for any $\Bbbk G$-module N. Our methods are more direct; we compute an explicit relative projective resolution for each N. Of course we are helped enormously by the fact that the representations of G are completely classified. Our first main result is:

Theorem 1. *Let M be an indecomposable $\Bbbk G$-module, which is not projective relative to χ. Then we have*

$$\Omega_\chi(M) \cong \Omega^{-2}(M)$$

if M has odd dimension, and

$$\Omega_\chi(M) \cong M$$

otherwise.

The ring structure of $H^*_\chi(G, \Bbbk)$ was not considered in [9]. Note, however, that if χ' is a subset of χ with size 2, then all products in $H^*_{\chi'}(G, \Bbbk)$ are zero, by a special case of Blowers' result. It is perhaps not surprising, therefore, that we have

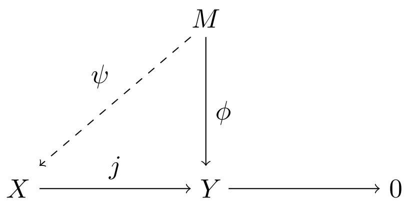
Theorem 2. *Let $\alpha_1, \alpha_2 \in H^*_\chi(G, \Bbbk)$, where both have strictly positive degree. Then $\alpha_1 \smile \alpha_2 = 0$.*

This paper is organised as follows. In section 2 we define relative projectivity and derive the results we will need to do the computations in later sections. This section follows [8, Section 2] fairly closely. As most proofs can be constructed by adapting familiar results projectivity to the relative case, they are omitted. In section 3 we describe the classification of modules for the Klein-four group and prove Theorem 1. We also compute $H^i_\chi(G, N)$ for every $\Bbbk G$-module N and prove Theorem 2.

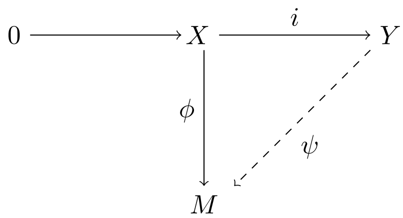
1.1. **Notation.** All groups under consideration are finite groups, and for any group G, by a $\Bbbk G$-module we mean a finitely-generated $\Bbbk$-vector space with compatible G action. The one-dimensional trivial $\Bbbk G$-module will be denoted by $\Bbbk_G$ or simply $\Bbbk$ when the group acting is obvious, and for $n \in \mathbb{N}$ and M a $\Bbbk G$-module we write nM for the direct sum of n copies of M.

2. Relative projectivity

In this section, let $p > 0$ be a prime and let G be a finite group of order divisible by p. Let $\Bbbk$ be a field of characteristic p and let χ be a set of subgroups of G. Now let M be a finitely generated $\Bbbk G$-module. M is said to be *projective relative to* χ if the following holds: let $\phi : M \to Y$ be a $\Bbbk G$-homomorphism and $j : X \to Y$ a surjective $\Bbbk G$-homomorphism which splits on restriction to any subgroup of $H \in \chi$. Then there exists a $\Bbbk G$-homomorphism ψ making the following diagram commute.



Dually, one says that M is *injective relative to* χ if the following holds: given an injective $\Bbbk G$-homomorphism $i : X \to Y$ which splits on restriction to each $H \in \chi$ and a $\Bbbk G$-homomorphism $\phi : X \to M$, there exists a $\Bbbk G$-homomorphism ψ making the following diagram commute.



These notions are equivalent to the usual definitions of projective and injective $\Bbbk G$-modules when we take $\chi = \{1\}$. We will say a $\Bbbk G$-homomorphism is χ-split if it splits on restriction to each $H \in \chi$. Since a $\Bbbk G$-module is projective relative to H if and only if it is also projective relative to the set of all subgroups of H, we usually assume χ is closed under taking subgroups.

We denote the set of G-fixed points in M by M^G. For any $H \leq G$ there is a $\Bbbk G$-map $M^H \to M^G$ defined as follows:

$$\mathrm{Tr}_H^G(x) = \sum_{\sigma \in S} \sigma x$$

where $x \in M$ and S is a left-transversal of H in G. This is called the relative trace or transfer. It is clear that the map is independent of the choice of S. If $H = 1$ we usually write this as Tr^G and call it simply the trace or transfer. For any set of subgroups χ of G we define the subspace

$$M^{G,\chi} := \sum_{H \in \chi} \mathrm{Tr}_G^H(M^H)$$

and quotient

$$M_\chi^G := \frac{M^G}{M^{G,\chi}}.$$

Now let N be another $\Bbbk G$-module. We can define an action of G on $\mathrm{Hom}_\Bbbk(M, N)$:

$$(g \cdot \phi)(x) = g\phi(g^{-1}x) \text{ for } g \in G, x \in M.$$

Notice that with this action we have $\mathrm{Hom}_\Bbbk(M, N)^G = \mathrm{Hom}_{\Bbbk G}(M, N)$. Further, the transfer construction gives a map

$$\mathrm{Tr}_H^G : \mathrm{Hom}_{\Bbbk H}(M, N) \to \mathrm{Hom}_{\Bbbk G}(M, N).$$

There a various ways to characterize relative projectivity:

Proposition 3. *Let G be a finite group of order divisible by p, χ a set of subgroups of G and M a $\Bbbk G$-module. Then the following are equivalent:*

(i) *M is projective relative to χ;*
(ii) *Every χ-split epimorphism of $\Bbbk G$-modules $\phi : N \to M$ splits;*
(iii) *M is injective relative to χ;*
(iv) *Every χ-split monomorphism of $\Bbbk G$-modules $\phi : M \to N$ splits;*
(v) *M is a direct summand of $\oplus_{H \in \chi} M \downarrow_H \uparrow^G$;*
(vi) *M is a direct summand of a direct sum of modules induced from subgroups in χ*
(vii) *There exists a set of homomorphisms $\{\beta_H : H \in \chi\}$ such that $\beta_H \in \mathrm{Hom}_{\Bbbk H}(M, M)$ and $\sum_{H \in \chi} \mathrm{Tr}_H^G(\beta_H) = \mathrm{id}_M$.*

The last of these is called *Higman's criterion.*

Proof. The proof when χ consists of a single subgroup of G can be found in [2, Proposition 3.6.4]. This can easily be generalised. □

For homomorphisms $\alpha \in \mathrm{Hom}_{\Bbbk G}(M, N)$ we have the following:

Lemma 4. *Let M, N be $\Bbbk G$-modules, χ a collection of subgroups of G, and $\alpha \in \mathrm{Hom}_{\Bbbk G}(M, N)$. Then the following are equivalent:*

(i) *α factors through $\oplus_{H \in \chi} M \downarrow_H \uparrow^G$.*
(ii) *α factors through some module which is projective relative to χ.*
(iii) *There exist homomorphisms $\{\beta_H \in \mathrm{Hom}_{\Bbbk H}(M, N) : H \in \chi\}$ such that $\alpha = \sum_{H \in \chi} \mathrm{Tr}_H^G(\beta_H)$.*

Proof. This is easily deduced from [2, Proposition 3.6.6]. □

The above tells us that $\mathrm{Hom}_\Bbbk(M, N)^{G,\chi}$ consists of the $\Bbbk G$-homomorphsims which factor through a module which is projective relative to χ. We write

$$\underline{\mathrm{Hom}}_{\Bbbk G}^\chi(M, N) := \mathrm{Hom}_\Bbbk(M, N)_\chi^G.$$

Let M be a $\Bbbk G$-module and let X be a $\Bbbk G$-module that is projective relative to χ. It is easily shown, using Proposition 3, that $M \otimes X$ is projective relative to χ. For example, the module $M \otimes X$ where $X = \bigoplus_{H \in \chi} \Bbbk_H \uparrow^G$ is projective relative to χ. Moreover, the natural map $\sigma : M \otimes X \to M$ given by

$$\sigma(m \otimes x) = m$$

is a χ-split $\Bbbk G$-epimorphism (to see the splitting, use the Mackey Theorem). It follows that for each M, there exists a $\Bbbk G$-module Q_0 which is projective relative to χ and a χ-split $\Bbbk G$-epimorphism $\pi_0 : Q_0 \to M$.

Let $\pi_0 : Q_0 \to M$ and $\pi_0' :\to Q_0' \to M$ be two such pairs. The proof of Schanuel's Lemma (see [2, Lemma 1.5.3, Lemma 3.9.1]) extends more or less verbatim to the relative case; if $K_0 = \ker(\pi)$ and $K_0' = \ker(\pi_0')$ then $K_0 \oplus Q_0' \cong K_0' \oplus Q_0$.

If we choose among all such pairs, one in which the dimension of Q_0 is minimal, the kernel K_0 is defined uniquely. This pair (Q_0, π_0) is called the relative projective cover of M. For this choice we set $\Omega_\chi(M) = K_0$. We can interate this construction, setting $\Omega_\chi^i(M) = \Omega_\chi(\Omega_\chi^{i-1}(M))$. Minimality implies that if K_0' is the kernel of any other χ-split $\Bbbk G$-epimorphism $Q_0' \to M$, then $K_0 \cong \Omega_\chi(M) \oplus (rel.proj)$, where (rel. proj) is some module which is projective relative to χ.

Dually, we always have that M is a submodule of $M \otimes X$ with $X = \bigoplus_{H \in \chi} \Bbbk_H \uparrow^G$, and the inclusion $\rho : M \to M \otimes X$ splits on restriction to each $H \in \chi$. It follows that for each M, there exists a $\Bbbk G$-module J_0 and a χ-split $\Bbbk G$-monomorphism $\rho_0 : M \to J_0$.

Let $\rho_0 : M \to J_0$ and $\rho_0' : M \to J_0'$ be two such pairs. Again. by the relative version of Schanuel's Lemma, if $C_0 = \operatorname{coker}(\pi)$ and $C_0' = \operatorname{coker}(\pi_0')$ then $C_0 \oplus J_0' \cong C_0' \oplus J_0$.

If we choose among all such pairs, one in which the dimension of J_0 is minimal, the cokernel C_0 is defined uniquely. The pair (J_0, ρ_0) is called a relative injective hull of M with respect to χ. For this choice we set $\Omega_\chi^{-1}(M) = K_0$. We can interate this construction, setting $\Omega_\chi^{-i}(M) = \Omega_\chi^{-1}(\Omega_\chi^{-(i-1)}(M))$. Minimality implies that if K_0' is the kernel of any other χ-split $\Bbbk G$-monomorphism $M \to J_0$, then $K_0' \cong \Omega_\chi^{-1}(M) \oplus (rel.proj)$, where (rel. proj) is some module which is projective relative to χ.

The following gives some properties of the operators Ω_χ^i.

Proposition 5. *Let M_1, M_2 be $\Bbbk G$-modules without summands which are projective relative to χ, and i, j nonzero integers. Then:*

(i) $\Omega_\chi^i(M_1 \oplus M_2) \cong \Omega_\chi^i(M_1) \oplus \Omega_\chi^i(M_2)$;
(ii) $\Omega_\chi^i(M)^* \cong \Omega_\chi^{-i}(M^*)$;
(iii) $M \cong \Omega_\chi(\Omega_\chi^{-1}(M)) \oplus$ *(rel. proj)* $\cong \Omega_\chi^{-1}(\Omega_\chi(M)) \oplus$ *(rel. proj.).*

Proof. (i) is obvious. (ii,iii) are easily deduced from the relative version of Schanuel's Lemma. □

(i) above shows that Ω_χ^i is a well-defined operator on the set of indecomposable $\Bbbk G$-modules which are not relatively projective to χ. Note that (iii) does not say that $\Omega_\chi \circ \Omega_\chi^{-1}$ is the identity in general. If we define $\Omega_\chi^0(M)$ to be the direct sum of all summands of M which are not projective relative to χ, then we have $\Omega^{i+j} = \Omega_\chi^i \circ \Omega_\chi^j$ for all i and j.

The following result is sometimes useful.

Lemma 6. *Let M be a $\Bbbk G$-module which is projective relative to a set χ of subgroups of G. Then $M^G = \sum_{H \in \chi} \operatorname{Tr}_H^G(M^H)$.*

Proof. See [8, Lemma 2.9] □

As a consequence of the above, if $M = N\oplus$ (rel. proj.), we get that $M_\chi^G = N_\chi^G$. The operators Ω_χ^i extend in a natural way to homomorphisms between modules. Let $f \in \mathrm{Hom}_{\Bbbk G}(M,N)$. Let $(Q,\pi),(Q',\pi')$ be the relative projective covers of M,N. Then the relative projectivity of Q ensures the existence of a homomorphism $\bar{f} \in \mathrm{Hom}_{\Bbbk G}(Q,Q')$ making the following diagram commute

$$\begin{array}{ccccccc}
\Omega_\chi(M) & \longrightarrow & Q & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \Omega_\chi(f)} & & \Big\downarrow{\scriptstyle \bar{f}} & & \Big\downarrow{\scriptstyle f} & & \\
\Omega_\chi(N) & \longrightarrow & Q' & \xrightarrow{\pi'} & N & \longrightarrow & 0
\end{array}$$

and an easy diagram chase shows that the image of $\Omega_\chi(f) : \bar{f}|_{\ker(\pi)}$ is contained in $\ker(\pi')$. In this way, f induces a homomorphism $\Omega_\chi(f) \in \mathrm{Hom}_{\Bbbk G}(\Omega_\chi(M),\Omega_\chi(N))$.

In a similar fashion, let $(J,\rho),(J',\rho')$ be the relative injective hulls of M,N respectively. Then relative injectivity of J' ensures the existence of a homomorphism $\tilde{f} \in \mathrm{Hom}(J,J')$ making the following diagram commute,

$$\begin{array}{ccccccc}
\Omega_\chi(M) & \longleftarrow & Q & \xleftarrow{\rho} & M & \longleftarrow & 0 \\
\Big\downarrow{\scriptstyle \Omega_\chi^{-1}(f)} & & \Big\downarrow{\scriptstyle \tilde{f}} & & \Big\downarrow{\scriptstyle f} & & \\
\Omega_\chi^{-1}(N) & \longleftarrow & Q' & \xleftarrow{\rho'} & N & \longleftarrow & 0
\end{array}$$

and a diagram chase shows that $\tilde{f}$ induces a well-defined homomorphism $\Omega_\chi^{-1}(f) \in \mathrm{Hom}(\Omega_\chi^{-1}(M),\Omega_\chi^{-1}(N))$. For a given homomorphism f, $\Omega_\chi^{-1}(\Omega_\chi(f))$ and $\Omega_\chi(\Omega_\chi^{-1}(f))$ are not equal to f in general, but $f - \Omega_\chi^{-1}(\Omega_\chi(f))$ and $f - \Omega_\chi(\Omega_\chi^{-1}(f))$ factor through a module which is projective relative to χ. By the discussion following Lemma 4, we have

Proposition 7. *For all $i \in \mathbb{Z}$, $\Omega_\chi^i(-)$ induces an isomorphism*

$$\underline{\mathrm{Hom}}_{\Bbbk G}^{\chi}(M,N) \cong \underline{\mathrm{Hom}}_{\Bbbk G}^{\chi}(\Omega_\chi^i(M),\Omega_\chi^i(N)).$$

As explained in the introduction, the idea of a relatively projective cover can be extended to a relatively projective resolution; that is, an exact complex

$$\ldots \to Q_i \to Q_{i-1} \to \ldots \to Q_0 \to M \to 0 \tag{3}$$

of relatively projective modules in which the connecting homomorphisms split over χ. If

$$\ldots \to Q'_i \to Q'_{i-1} \to \ldots \to Q'_0 \to M \to 0 \tag{4}$$

is another relatively projective resolution, then it turns out that any two chain maps between them are chain homotopic (see [2, Theorem 3.9.3] for the version with χ consisting of one subgroup - the proof of the more general version is the same). Consequently, for any $\Bbbk G$-module N, the homology groups of the induced complex

$$0 \to \mathrm{Hom}_{\Bbbk G}(Q_0, M) \to \ldots \to \mathrm{Hom}_{\Bbbk G}(Q_i, M) \to \ldots$$

are independent of the choice of resolution. The homology groups of this complex are by definition the relative Ext-groups $\mathrm{Ext}^i_{\Bbbk G,\chi}(M,N)$. The relative cohomology of G with respect to χ with coefficients in N is the special case

$$H^i_\chi(G,N) := \mathrm{Ext}^i_{\Bbbk G,\chi}(\Bbbk, N).$$

We will use a minimal relative projective resolution of the trivial module to compute relative cohomology; that is, a relatively projective resolution

$$\ldots \to Q_i \overset{\partial_{i-1}}{\to} Q_{i-1} \to \ldots \overset{\partial_0}{\to} Q_0 \to \Bbbk \to 0. \tag{5}$$

in which $\ker(\partial_{i-1}) = \Omega^i_\chi(\Bbbk)$. We can construct this by taking for each i a short exact sequence

$$0 \to \Omega^{i+1}_\chi(\Bbbk) \overset{\rho_i}{\to} Q_i \overset{\pi_i}{\to} \Omega^i_\chi(\Bbbk) \to 0$$

and setting $\partial_i := \rho_i \pi_{i+1}$. For each i let

$$\delta_i : \mathrm{Hom}_{\Bbbk G}(Q_i, \Bbbk) \to \mathrm{Hom}_{\Bbbk G}(Q_{i+1}, \Bbbk)$$

denote the map induced by ∂_i.

Our main tool will be as following:

Proposition 8. *Let N be a $\Bbbk G$-module. Then we have*
(i) $H^0_\chi(G,N) = N^G$;
(ii) $H^i_\chi(G,N) \cong \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Omega^i_\chi(\Bbbk), N)$.

The proof is the same as in the case $\chi = \{1\}$, but we give a sketch for lack of a good reference to this proof.

Proof. We first show that for each $i \geq 0$,

$$\ker(\delta_i) \cong \mathrm{Hom}_{\Bbbk G}(\Omega^i_\chi(\Bbbk), N).$$

To see this, let $\phi \in \ker(\delta_i) \subseteq \mathrm{Hom}_{\Bbbk G}(Q_i, N)$. For $x \in \Omega^i_\chi(\Bbbk)$, choose $q \in Q_i$ such that $\pi_i(q) = x$ and define $\hat{\phi}(x) = \phi(q)$. Then $\hat{\phi} \in \mathrm{Hom}_{\Bbbk G}(\Omega^i_\chi(\Bbbk, N))$. The assignment $\phi \to \hat{\phi}$ is well-defined: for if $q' \in Q_i$ with $\pi_i(q') = x$ and $\tilde{\phi}(x) := \phi(q')$, then since $q - q' \in \ker(\pi_i)$ we get $q - q' \in \mathrm{im}(\partial_i)$ and $\phi(q-q') = 0$ since $\phi \in \ker(\delta_i)$. Conversely, given $\phi \in \mathrm{Hom}_{\Bbbk G}(\Omega^i_\chi(\Bbbk), N)$ we can define $\hat{\phi} = \phi \circ \pi_i \in \ker(\delta_i)$. It's easy to see that the two assignments are inverse to each other.

This in particular shows that (i) holds, since $\mathrm{Hom}_{\Bbbk G}(\Bbbk, N) \cong N^G$. We now show that $\mathrm{im}(\delta_{i-1})$ consists of the homomorphisms in $\mathrm{Hom}_{\Bbbk G}(\Omega^i(\Bbbk), N)$ which factor through a module which is projective relative to χ. To see this, first suppose $\phi \in \mathrm{im}(\delta_{i-1}) \subseteq \mathrm{Hom}_{\Bbbk G}(Q_i, N)$, say $\phi = \psi \circ \partial_{i-1}$ where $\psi \in \mathrm{Hom}_{\Bbbk G}(Q_{i-1}, N)$. Then with $x \in \Omega^i_\chi(\Bbbk)$ and $q, \hat{\phi}$ as before we note that

$$\psi \circ \rho_{i-1}(x) = \psi \circ \rho_{i-1} \circ \pi_i(q) = \psi \circ \partial_i(q) = \phi(q) = \hat{\phi}(x)$$

which shows that $\hat{\phi}$ factors through the module Q_{i-1} which is projective relative to χ. Conversely, if $\phi \in \mathrm{Hom}_{\Bbbk G}(\Omega^i_\chi, \Bbbk)$ factors through any module which is projective relative to χ, then it factors through Q_{i-1}, because ρ_{i-1} is injective and Q_{i-1} is also an injective module with respect to χ by Lemma 3.

□

One can define a pairing $\smile : H^i_\chi(G,\Bbbk) \otimes H^j_\chi(G,\Bbbk) \to H^{i+j}_\chi(G,\Bbbk)$ in a few different ways. On the one hand, elements of $H^*_\chi(G,\Bbbk) = \mathrm{Ext}^*_{\Bbbk G,\chi}(\Bbbk,\Bbbk)$ can be viewed as equivalence classes of extensions of $\Bbbk$ by $\Bbbk$ split over χ, and the usual Yoneda splice gives the required pairing; see [2, Section 2.6,3.9] for details in the case χ consisting of only one subgroup. Some other constructions in the case $\chi = \{1\}$ are given in [6], and all of these extend in a natural way to arbitrary χ. Happily, all these methods give the same construction. In the present article we will use the following construction: recall that

$$H^i_\chi(G, \Bbbk) \cong \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Omega^i_\chi(\Bbbk), \Bbbk).$$

Similarly

$$H^j_\chi(G, \Bbbk) = \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Omega^j_\chi(\Bbbk), \Bbbk) \cong \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Omega^{i+j}_\chi(\Bbbk), \Omega^i_\chi(\Bbbk))$$

with the second isomorphism arising from Proposition 7. Therefore we may define a product as follows: for $\alpha \in H^i_\chi(G, \Bbbk)$ and $\beta \in H^j_\chi(G, \Bbbk)$ choose $f \in \mathrm{Hom}_{\Bbbk G}(\Omega^i_\chi(\Bbbk), \Bbbk)$, $g \in \mathrm{Hom}_{\Bbbk G}(\Omega^j_\chi(\Bbbk), \Bbbk)$ respresenting α, β respectively. Then $\Omega^i_\chi(g) \in \mathrm{Hom}_{\Bbbk G}(\Omega^{i+j}_\chi(\Bbbk), \Omega^i_\chi(\Bbbk))$, so that

$$f \circ \Omega^i_\chi(g) \in \mathrm{Hom}_{\Bbbk G}(\Omega^{i+j}_\chi(\Bbbk), \Bbbk).$$

We take $\alpha \smile \beta$ to be the cohomology class represented by $f \circ \Omega^i_\chi(g)$. This is called the *cup product* of α and β.

3. Representations of $C_2 \times C_2$

In this section, let $G = \langle \sigma, \tau \rangle$ denote the Klein four-group, and let $\Bbbk$ be a field of characteristic 2 (not necessarily algebraically closed). We set $\chi = \{H_1, H_2, H_3\}$, the set of all proper nontrivial subgroups of G, where $H_1 = \langle \sigma \rangle, H_2 = \langle \tau \rangle, H_3 = \langle \sigma\tau \rangle$.

Let $X := \sigma - 1 \in \Bbbk G$, $Y := \tau - 1 \in \Bbbk G$. Then $X^2 = Y^2 = 0$, $\Bbbk G$ is isomorphic to the quotient ring

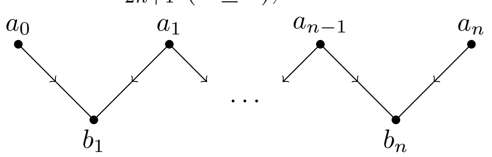
$$R := \Bbbk[X, Y]/(X^2, Y^2),$$

and $\Bbbk G$-modules can be viewed as R modules. We will describe R-modules by means of the diagrams for modules first introduced by Alperin in [1]. In these diagrams, nodes represent basis elements, and two nodes labelled a and b are joined by a south-west directed arrow if $Xa = b$, and by a south-east directed arrow if $Ya = b$. If no south-west arrow begins at a then it is understood that $Xa = 0$, similarly for Y.

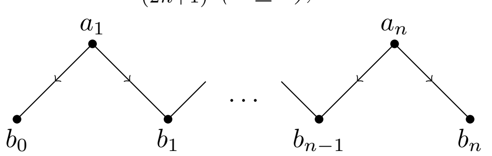
Our statement of the classification of $\Bbbk G$-modules resembles that found in [7] and we recommend this reference as an easily accessible proof.

Proposition 9. *Let M be an indecomposable $\Bbbk G$-module. Then M is isomorphic to one of the following:*

(1) *The module V_{2n+1} $(n \geq 0)$, with odd dimension $2n + 1$ and diagram*

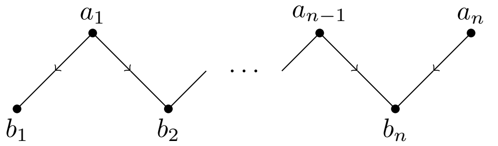


(2) *The module $V_{-(2n+1)}$ $(n \geq 0)$, with odd dimension $2n + 1$ and diagram*

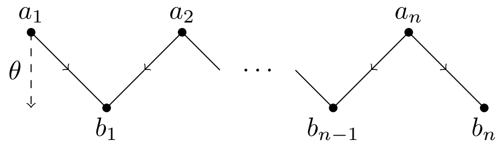


Note that $V_1 \cong V_{-1} \cong \Bbbk$, with trivial G-action, but otherwise these modules are pairwise non-isomorphic.

(3) *The module* $V_{2n,\infty}$*,* $(n \geq 1)$*, with even dimension* $2n$ *and diagram*

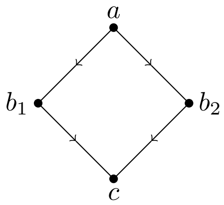


(4) *The module* $V_{2n,\theta}$*,* $(n \geq 1)$*, with even dimension* $2n$ *and diagram,*



Here, $\theta(x) = \sum_{i=0}^{n} \lambda_i x^{n-i}$ *is a power of an irreducible monic polynomial with coefficients in* $\Bbbk$ *and the dotted line labelled by* θ *indicates that* $Xa_1 = \sum_{i=1}^{n} \lambda_i b_i$*.*

(5) *The projective indecomposable module* P*, with dimension 4 and diagram*



The following, also taken from [7], may be proved directly from the classification above.

Proposition 10. *Let* M *be an indecomposable* $\Bbbk G$*-module. Then we have*

(1) $M \cong M^*$ *if* M *is even-dimensional.*
(2) $M^* \cong V_{-(2n+1)}$ *if* $M \cong V_{2n+1}$ *is odd dimensional.*
(3) $M^* \cong V_{2n+1}$ *if* $M \cong V_{-(2n+1)}$ *is odd-dimensional.*

Clearly (3) follows from (2) above, but we include it for completeness. In addition,

Proposition 11. *Let* M *be an indecomposable* $\Bbbk G$*-module. Then we have*

(1) $\Omega(M) \cong M$ *if* M *is even-dimensional.*
(2) $\Omega^{-1}(M) \cong V_{-(2n+3)}$ *if* $M \cong V_{-(2n+1)}$ *is odd dimensional.*
(3) $\Omega(M) \cong V_{2n+3}$ *if* $M \cong V_{2n+1}$ *is odd-dimensional.*

Again (3) follows from (2) when we take into account that $\Omega(M)^* \cong \Omega^{-1}(M^*)$ in general.

3.1. **Relative shifts.** The goal of this subsection is to prove Theorem 1.

Among the indecomposable $\Bbbk G$-modules listed in the previous section, only four are projective relative to χ. These are the projective indecomposable P, and the three modules $V_{2,\infty}$, $V_{2,x}$ and $V_{2,x+1}$. Here the last two are the indecomposable modules $V_{2,\theta}$ where $\theta(x)$ is the monic irreducible x or $x+1 \in \Bbbk[x]$. Note that τ acts trivially on $V_{2,\infty} = \Bbbk_{H_2} \uparrow^G$, while σ acts trivially on $V_{2,x} = \Bbbk_{H_1} \uparrow^G$ and $\sigma\tau$ acts trivially on $V_{2,x+1} = \Bbbk_{H_3} \uparrow^G$. As these three play in important role in what follows, we denote them by Q_τ, Q_σ and $Q_{\sigma\tau}$ respectively. We set $Q = Q_\sigma \oplus Q_\tau \oplus Q_{\sigma\tau}$.

We begin by considering odd-dimensional modules.

Lemma 12. *Let* $n \geq 0$*:*

(1) *The relative projective cover of* $V_{-(2n+1)}$ *is* $Q \oplus nP$*.*

(2) *We have* $\Omega_\chi(V_{-(2n+1)}) \cong V_{-(2n+5)}$.

Proof. Let $M \cong V_{-(2n+1)}$ and let $\pi : N \to M$ be its relative projective cover with respect to χ. N must decompose as a direct sum of modules of the form P, Q_σ, Q_τ and $Q_{\sigma\tau}$.

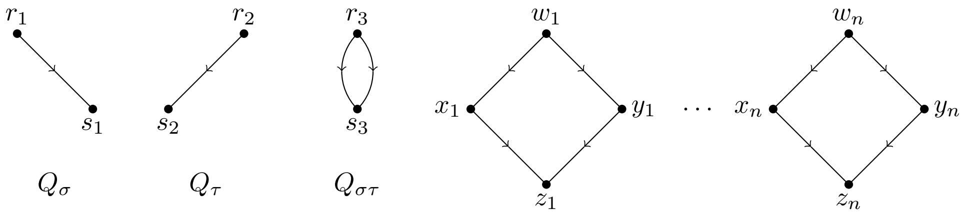
Let $a_1, a_2, \dots, a_n, b_0, b_1, \dots, b_n$ be a basis of M, with action given by the diagram as in Proposition 9. Since π is a surjective $\Bbbk G$-map and no a_i is fixed by any element of G, the same must be true of their unique pre-images. The modules Q_σ, Q_τ and $Q_{\sigma\tau}$ all have non-trivial kernels. Therefore N contains at least n copies of P.

On the other hand, we have, for any i,

$$M \downarrow_{H_i} \cong \Bbbk_{H_i} \oplus n\Bbbk H_i \tag{6}$$

The restrictions to H_1 of P, Q_τ and $Q_{\sigma\tau}$ contain no trivial H_1-summands. So N must contain a direct summand isomorphic to Q_σ if π is to split on restriction to H_1. A similar argument (restricting to H_2, H_3) shows that N must contain summands isomorphic to Q_τ and $Q_{\sigma\tau}$.

We will construct a surjective $\Bbbk G$-homomorphism $Q \oplus nP \to M$. The following diagrams label the basis elements:



(the diagram for $Q_{\sigma\tau}$ is not as described in Proposition 9, but self-explanatory.) We now define a linear map $\pi : Q \oplus nP \to M$ by

- $\pi(w_i) = a_i$ for $i = 1, \dots, n$.
- $\pi(x_i) = b_{i-1}$ for $i = 1, \dots, n$.
- $\pi(y_i) = b_i$ for $i = 1, \dots, n$.
- $\pi(z_i) = 0$ for $i = 1, \dots, n$.
- $\pi(s_i) = 0$ for $i = 1, 2, 3$.
- $\pi(r_1) = \pi(r_3) = a_0$.
- $\pi(r_2) = a_n$.

The reader should check that π is a $\Bbbk G$-homomorphism. The kernel of π is spanned by

$$\{z_i : i = 1, \dots, n\} \cup \{s_1, s_2, s_3\} \cup \{x_i + y_{i-1} : i = 2, \dots, n\} \cup \{x_1 + r_1, x_1 + r_3, y_n + r_2\}.$$

It has dimension $2n + 5$, and the fixed-point space within this module is spanned by $\{z_1, z_2, \dots, z_n, s_1, s_2, s_3\}$, so it has dimension $n + 3$. It is easily checked that no element of the kernel outside of the fixed-point space is fixed by any subgroup H_i. Therefore

$$\ker(\pi) \downarrow_{H_i} \cong \Bbbk_{H_i} \oplus (n+2)\Bbbk H_i$$

for any i. This, combined with (6) and the fact that

$$(Q \oplus nP) \downarrow_{H_i} \cong 2\Bbbk_{H_i} \oplus (2n+2)\Bbbk H_i$$

shows that π splits on restriction to any H_i. The construction ensures the minimality of $Q \oplus nP$, so $Q \oplus nP = N$, proving (1). Further, $\Omega_\chi(M) = \ker(\pi)$, and the classification of $\Bbbk G$-modules, together with the fact that $\ker(\pi)$ must be indecomposable, implies that $\ker(\pi) \cong V_{-(2n+5)}$, proving (2). □

The following follows immediately the above using Propositions 10 and 5(3).

Lemma 13. *Let* $n \geq 0$*: Then we have* $\Omega_\chi(V_{(2n+5)}) \cong V_{(2n+1)}$.

To complete the picture for odd-dimensional modules, it remains only to show that

Lemma 14. *Let $M \cong V_3$. Then:*

1. *The relative projective cover of M is Q;*
2. *We have $\Omega_\chi(M) \cong V_{-3}$.*

Proof. We have $M \downarrow_{H_i} \cong \Bbbk_{H_i} \oplus \Bbbk H_i$, for $i = 1, 2, 3$, so once more the projective cover must contain a summand isomorphic to Q. We shall construct a $\Bbbk G$-homomorphism $\pi : Q \to M$. We retain the notation for a basis of Q used in Lemma 12; a basis for M is $\{a_0, a_1, b_1\}$ with action given as in the classification.

Define:

- $\pi(r_1) = a_0$
- $\pi(r_2) = a_1$
- $\pi(r_3) = a_0 + a_1$.
- $\pi(s_1) = \pi(s_2) = \pi(s_3) = b_1$.

The reader should check this is a $\Bbbk G$-homomorphism. The kernel of π is spanned by $\{s_1 + s_2, s_2 + s_3, r_1 + r_2 + r_3\}$, and the fixed-point space of the kernel is two-dimensional, spanned by $\{s_1 + s_3, s_2 + s_3\}$. Noting that

$$X(r_1 + r_2 + r_3) = s_2 + s_3, Y(r_1 + r_2 + r_3) = s_1 + s_3,$$

we see that the kernel of π is indecomposable, and as a $\Bbbk G$-module is isomorphic to V_{-3}. Therefore

$$\ker(\pi)_{H_i} \oplus \Bbbk_{H_i} \oplus \Bbbk H_i$$

for all i, from which we deduce that π splits on restriction to each H_i. Our construction ensures the minimality of Q, so Q is indeed the relative projective cover of M, proving (1), and $\ker(\pi) = \Omega_\chi(M) \cong V_{-3}$, proving (2). □

We now turn to even dimensional modules. Note that $V_{2,\infty} = Q_\tau$ is already projective relative to χ, so $\Omega_\chi(V_{2,\infty})$ is not defined.

Lemma 15. *Let $n \geq 2$ and $M \cong V_{2n,\infty}$. Then:*

1. *The relative projective cover of M is $2Q_\tau \oplus (n-1)P$;*
2. *We have $\Omega_\chi(M) \cong M$.*

Proof. Let $\pi : N \to M$ be the relative projective cover of M. Notice that

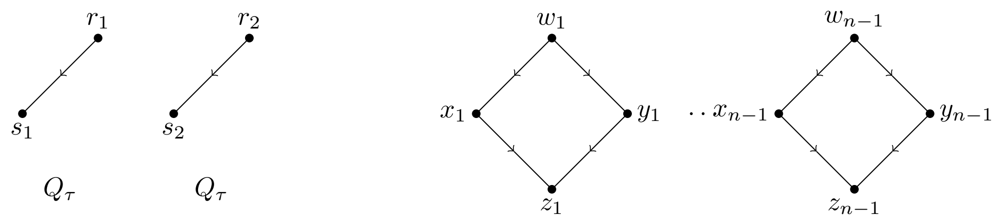
$$M \downarrow_{H_i} = n \Bbbk H_i \qquad (7)$$

for $i = 1, 3$ whereas

$$M \downarrow_{H_2} = 2\Bbbk_{H_2} \oplus (n-1)\Bbbk H_2. \qquad (8)$$

So if $\pi : N \to M$ is to split on restriction to H_2, N must contain a pair of direct summands isomorphic to Q_τ. On the other hand, retaining the notation from Proposition 9, the basis elements $a_1, \ldots, a_{n-1}$ are not fixed by any element of G, so the same must be true of their unique pre-images in N. From this it follows that N must contain $n-1$ direct summands isomorphic to P.

We will construct a $\Bbbk G$-homomorphism $2Q_\tau \oplus (n-1)P \to M$. The following diagram gives the labelling for a basis of the domain:



We define:

- $\pi(w_i) = a_i$ for $i = 1, \ldots, n-1$.
- $\pi(x_i) = b_i$ for $i = 1, \ldots, n-1$.
- $\pi(y_i) = b_{i+1}$ for $i = 1, \ldots, n-1$.
- $\pi(z_i) = 0$ for $i = 1, \ldots, n-1$.
- $\pi(r_1) = b_1$.
- $\pi(s_1) = 0$.
- $\pi(r_2) = a_n$.
- $\pi(s_2) = b_n$.

The reader should check that π is a $\Bbbk G$-homomorphism. The kernel of π is spanned by

$$\{z_i : i = 1, \ldots, n-1\} \cup \{x_i + y_{i-1} : i = 2, \ldots, n-1\} \cup \{s_1, x_1 + r_2, y_{n-1} + s_2\}.$$

This has dimension $2n$. The fixed points within this module are spanned by

$$\{z_i : i = 1, \ldots, n-1\} \cup \{s_1\}.$$

These span the fixed points of H_1 and H_3, while H_2 has a fixed point space of dimension $n+1$, spanned by the above and $y_{n+1} + s_2$. Therefore we have

$$\ker(\pi) \downarrow_{H_i} \cong n\Bbbk H_i$$

for $i = 1, 3$ and

$$\ker(\pi) \downarrow_{H_2} \cong 2\Bbbk_{H_2} \oplus (n-1)\Bbbk H_2.$$

Note that

$$(2Q_\tau \oplus (n-1)P) \downarrow_{H_i} \cong 2n\Bbbk H_i$$

for $i = 1, 3$ and

$$(2Q_\tau \oplus (n-1)P) \downarrow_{H_2} \cong 4\Bbbk_{H_2} \oplus (2n-2)\Bbbk H_i.$$

Thus, π splits on restriction to each H_i. The construction ensures the minimality of $2Q_\tau \oplus (n-1)P$, so this is equal to N and we have (1). Further, $\ker(\pi) = \Omega_\chi(M)$ must be indecomposable. By the classification (looking at the dimension of the fixed point space of each subgroup of G to distinguish among modules of even dimension) we must have $\Omega_\chi(M) \cong M$ as required for (2). □

Notice that if $\theta(x) = x^n$, then $V_{2n,\theta}$ can be obtained from $V_{2n,\infty}$ by applying the automorphism of G which swaps σ and τ. Similarly if $\theta(x) = (x+1)^n$, then $V_{2n,\theta}$ can be obtained from $V_{2n,\infty}$ by applying the automorphism of G which swaps $\sigma\tau$ and τ. We therefore obtain immediately from Lemma 15 above that $\Omega_\chi(M) = M$ if M is one of these.

It remains only to prove the following:

Lemma 16. *Let $n \geq 1$ and let $M \cong V_{2n,\theta}$, where θ is neither x^n nor $(x+1)^n$. Then:*

(1) *The relative projective cover of M is nP;*
(2) *$\Omega_\chi(M) \cong M$.*

Proof. Observe that $M \downarrow_{H_i} = n\Bbbk H_i$ for each i. The proof of [7, Proposition 3.1] shows that the projective (as opposed to relatively projective) cover of M is nP and $\Omega(M) \cong M$, so there is a surjective $\Bbbk G$-homomorphism $\pi : nP \to M$ with kernel isomorphic to M. Noting that $nP \downarrow_{H_i} \cong 2n\Bbbk H_i$ for each i, we see that π splits on restriction to each H_i. On the other hand, if N is a $\Bbbk G$-module having Q_τ (resp. $Q_\sigma, Q_{\sigma\tau}$) as a direct summand then $N \downarrow_{H_i}$ contains a pair of trivial $\Bbbk H_i$-modules as direct summand, and no surjective homomorphism $N \to M$ may split. This shows the minimality of the dimension of nP among relatively projective modules with a χ-split epimorphism to M, i.e. we have proved (1). We also have

$$\Omega_\chi(M) = \ker(\pi) = \Omega(M) \cong M$$

as required for (2). □

Remark 17. Combining all the Lemmas in this section with Proposition 11, we obtain Theorem 1.

3.2. **Computing Cohomology.** In this subsection we will determine $H^i(G,N)$ for all $i \geq 0$ and for all indecomposable $\Bbbk G$-modules N. First observe that if N is projective relative to χ, then $H^i(G,N) = 0$ for all $i > 0$: this is an immediate consequence of Proposition 8(ii). Further, recall from part (i) of the same that $H^0_\chi(G,N) = N^G$ for any $\Bbbk G$-module. It follows that:

Proposition 18. *Let $N \in \{P, Q_\sigma, Q_\tau, Q_{\sigma\tau}\}$. Then.*

$$\dim(H^i_\chi(G,N)) = \begin{cases} 1 & i = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Now we consider even-dimensional modules which are not relatively projective. Recall that for $i > 0$ we have

$$H^i_\chi(G,N) = \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Omega^i_\chi(\Bbbk), N) \cong \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Bbbk, \Omega^{-i}_\chi(N)) \cong \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Bbbk, N) \cong N^G_\chi$$

using the fact that, for these modules N, we have $\Omega^{-i}_\chi(N) \cong N$.

We obtain by direct calculation:

Proposition 19. *Let N be an even-dimensional $\Bbbk G$-module which is not projective relative to χ. Then.*

$$\dim(H^i_\chi(G,N)) = \begin{cases} n & i = 0 \\ n-1 & \textit{otherwise} \end{cases}$$

if $N \cong V_{2n,\infty}$ or $N \cong V_{2n,\theta}$ where $\theta(x) = x^n$ or $\theta(x) = (x+1)^n$, while

$$\dim(H^i_\chi(G,N)) = n$$

for any i, if $V \cong V_{2n,\theta}$ for some other choice of θ.

For odd-dimensional modules we proceed as follows. Let N be an odd-dimensional indecomposble module and let $i > 0$. Then

$$H^i_\chi(G,N) = \underline{\mathrm{Hom}}_{\Bbbk G}(\Omega^i_\chi(\Bbbk), N) \cong \underline{\mathrm{Hom}}_{\Bbbk G}(\Bbbk, \Omega^{-i}_\chi(N) \cong \underline{\mathrm{Hom}}_{\Bbbk G}(\Bbbk, \Omega^{2i}(N)) \cong \Omega^{2i}(N)^G_\chi$$

using Theorem 1. Suppose $N \cong V_{2n+1}$ where $n \geq 0$. Then $\Omega^{2i}(N) \cong V_{2(n+2i)+1}$. A basis for $V_{2(n+2i)+1}$ is given by $\{a_0, a_1, \ldots, a_{n+2i}, b_1, b_2, \ldots, b_{n+2i}\}$, with action given by the diagram in Proposition 9. The b_i are all fixed points, and in addition a_0 is fixed by H_1, a_{n+2i} by H_2 and $a_0 + a_1 + \ldots + a_{n+2i}$ by H_3. Therefore b_1, b_{n+2i} and $b_1 + b_2 + \ldots + b_{n+2i}$ lie in $\Omega^{2i}(N)^{G,\chi}$. We therefore have

Proposition 20. *Let $N \cong V_{2n+1}$ for some $n \geq 0$. Then*

1. $\dim(H^0_\chi(G,N)) = n$ *if $n > 0$, and 1 if $n = 0$.*
2. $\dim(H^i_\chi(G,N)) = \max(0, n + 2i - 3)$ *for $i > 0$.*

Remark 21. This includes [9, Theorem 1.2] as a special case ($n = 0$).

For the remaining odd dimensional modules things are a little more complicated, since $\Omega^{2i}(N)$ eventually moves into the "positive" part of the spectrum. We begin by noting that if $n \geq 0$, then $V^{H_i}_{-(2n+1)} = V^G_{-(2n+1)}$ for all i. Therefore $(V_{-(2n+1)})^{G,\chi} = 0$.

Now let $N \cong V_{-(2n+1)}$ where $n \geq 1$. For $i \leq n/2$ we have $\Omega^{2i}(N) \cong V_{-(2(n-2i)+1)}$. Therefore

$H^i_\chi(G,N) = \underline{\mathrm{Hom}}_{\Bbbk G}(\Omega^i_\chi(\Bbbk), N) \cong \underline{\mathrm{Hom}}_{\Bbbk G}(\Bbbk, \Omega^{-i}_\chi(N) \cong \underline{\mathrm{Hom}}_{\Bbbk G}(\Bbbk, \Omega^{2i}(N)) \cong \Omega^{2i}(N)^G$.

For $i > n/2$ we have $\Omega^{2i}(N) \cong V_{2(2i-n)+1}$. We therefore obtain the following:

Proposition 22. *Let $N \cong V_{-(2n+1)}$ where $n \geq 1$. Then*

$$\dim(H^i_\chi(G,N)) = \begin{cases} n+1-2i & i \leq n/2 \\ \max(0, 2i-n-3) & i > n/2. \end{cases}$$

3.3. **Calculating cup products.** The aim of this section is to prove Theorem 2. We begin with a lemma:

Lemma 23. *Let $M \cong V_{-(2m+1)}$ and $N \cong V_{-(2n+1)}$ for some $m > n \geq 0$. Let $\phi \in \mathrm{Hom}_{\Bbbk G}(M,N)$. Then*
(1) $\mathrm{im}(\phi) \subseteq N^G$;
(2) $M^G \subseteq \ker(\phi)$.

Proof. Note first that $\phi(M^G) \subseteq N^G$ for arbitrary G and $\Bbbk G$-modules M and N. Let $a_1, a_2, \ldots, a_m, b_0, b_1, \ldots, b_m$ and $a'_1, a'_2, \ldots, a'_n, b'_0, b'_1, \ldots, b'_n$ be bases of M and N respectively, with action given by the diagrams in proposition 9. Note that if $n = 0$, then (1) is immediate. So suppose $n > 0$ and (1) does not hold: then we can find a maximal $k \geq 1$ such that $\phi(a_k) \notin N^G$.

We claim that $k = m$. To see this, write

$$\phi(a_k) = \sum_{i=1}^n \lambda_i a'_i \mod N^G.$$

Then

$$\phi(b_k) = \phi(Ya_k) = Y\phi(a_k) = \sum_{i=1}^n \lambda_i b'_i.$$

If $k < m$ then also

$$\phi(b_k) = \phi(Xa_{k+1}) = X\phi(a_{k+1}) = 0$$

since $\phi(a_{k+1}) \in N^G$. So $\lambda_i = 0$ for all i and $\phi(a_k) \in N^G$, a contradiction.

Now we claim that, for all $0 \leq j \leq n$, we have

$$\phi(a_{m-j}) = \sum_{i=j+1}^n \lambda_i a'_{i-j} \mod N^G \tag{9}$$

and $\lambda_i = 0$ for $i = 1, \ldots, j$. We prove this by induction on j. The base case $j = 0$ is true by definition. Assuming the above for some $0 \leq j < n$ and noting that $n < m$, we have

$$\phi(b_{m-j-1}) = \phi(Xa_{m-j}) = X\phi(a_{m-j}) = \sum_{i=j+1}^n \lambda_i b'_{i-j-1}.$$

But

$$\phi(b_{m-j-1}) = \phi(Ya_{m-j-1}) = Y\phi(a_{m-j-1}) \in YN = \langle b'_1, \ldots, b'_n \rangle$$

which shows that $\lambda_{j+1} = 0$. Therefore

$$\phi(b_{m-j-1}) = \sum_{i=j+2}^n \lambda_i b'_{i-j-1}$$

which shows that

$$\phi(a_{m-j-1}) = \sum_{i=j+2}^n \lambda_i a'_{i-j-1} \mod N^G$$

proving our claim. Taking $j = n$ in (9) shows that $\phi(a_m) \in N^G$, a contradiction. This proves (1).

For (2), let $x \in M^G$. We may write

$$x = \sum_{i=0}^{m} \mu_i b_i$$

for some coefficients μ_i. Then

$$\phi(x) = \sum_{i=0}^{m} \mu_i \phi(b_i) = \mu_0 \phi(X a_0) + \sum_{i=1}^{m} \mu_i \phi(Y a_{i-1}) = \mu_0 X \phi(a_0) + Y \phi(\sum_{i=1}^{n} \mu_i a_i) = 0$$

by (1). □

The following is immediate:

Corollary 24. *Let $L \cong V_{-(2l+1)}$, $M \cong V_{-(2m+1)}$ and $N \cong V_{-(2n+1)}$ for some $l > m > n \geq 0$. Let $\phi \in \mathrm{Hom}_{\Bbbk G}(M, N)$ and $\psi \in \mathrm{Hom}_{\Bbbk G}(L, M)$. Then $\phi \circ \psi = 0$.*

We may now proceed with the proof of Theorem 2:

Proof. Let $i, j > 0$. Let $\alpha \in H^i_\chi(G, \Bbbk)$ and $\beta \in H^j_\chi(G, \Bbbk)$. Choose $\phi \in \mathrm{Hom}_{\Bbbk G}(\Omega^i_\chi(\Bbbk), \Bbbk)$ and $\psi \in \mathrm{Hom}_{\Bbbk G}(\Omega^j_\chi(\Bbbk), \Bbbk)$, such that the equivalence classes

$$[\phi] \in \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Omega^i_\chi(\Bbbk), \Bbbk), [\psi] \in \underline{\mathrm{Hom}}^\chi_{\Bbbk G}(\Omega^j_\chi(\Bbbk), \Bbbk)$$

represent α and β respectively. By definition, $\alpha \smile \beta$ is represented by $[\phi \circ \Omega^i_\chi(\psi)]$. By Lemma 12 we have

$$\phi \in \mathrm{Hom}(V_{-(2i+1)}, V_{(-1)}), \Omega^i_\chi(\psi) \in \mathrm{Hom}(V_{-(2i+2j+1)}, V_{-(2i+1)})$$

and by Corollary 24 the composition of these two is the trivial map. □

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