Middlesex University London

MSO4311 Week 15: Wiener Processes

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25th January 2021

The simple random walk we looked at in the last section was defined as

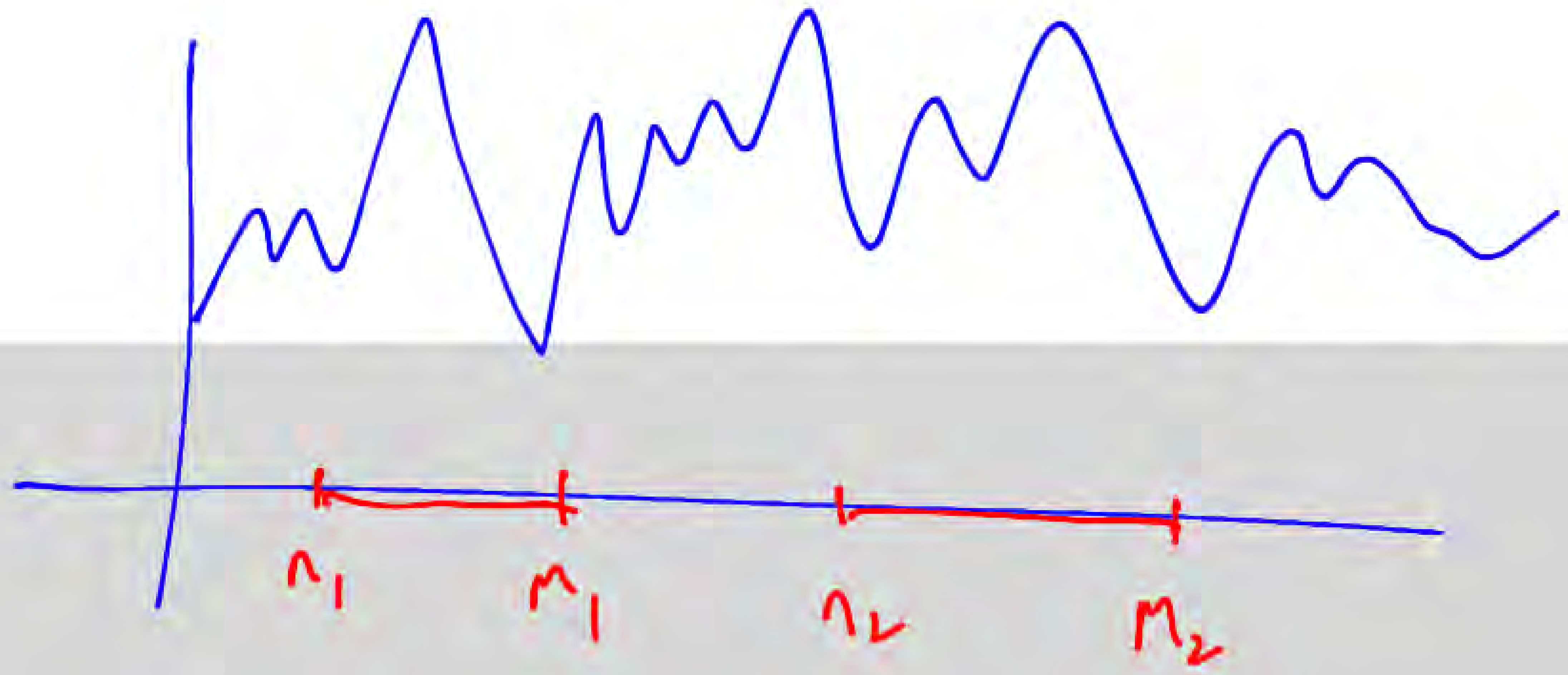
$$\boxed{S_n = X_1 + X_2 + \ldots + X_n}$$

$$X_i = \begin{cases} 1 & \text{with prob } 0.5 \\ -1 & \text{'' \quad '' \quad ''} \end{cases}$$

where X_i are iid Bernoulli trials with probability of success p.
We saw that when $p = 0.5$ this is a martingale.

walk[50:100] - walk[50]

1. The random variable $S_{m+n} - S_n$ has **the same distribution** as S_m. This doesn't depend on n so that at each stage it is like the random walk is beginning again. This is called **time-homogeneity** and is the basis of an important stochastic process called a Markov chain.
2. The random variable $S_{m_1} - S_{n_1}$ **is independent** of $S_{m_2} - S_{n_2}$ whenever $n_1 < m_1 < n_2 < m_2$, i.e. whenever the two intervals $[n_1, m_1]$ and $[n_2, m_2]$ are disjoint.



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Activity 14.11

▷ Task: Prove that the simple random walk has these properties.

① Time homogeneity.

$$P(S_{m+n} - S_n \le x) = P(\cancel{X_1} + \cancel{X_2} + \dots + X_{m+n} - (\cancel{X_1} + \cancel{X_2} + \dots + X_n) \le x)$$

$$= P(X_{n+1} + X_{n+2} + \dots + X_{n+m} \le x)$$

X_i are all iid r.v.s.

$$= P(X_1 + X_2 + \dots + X_m \le x)$$

X_{n+1} has same distribution as X_1

$$= P(S_m \le x)$$

2) $n_1 < m_1 < n_2 < m_2$ $\qquad S_{m_1} - S_{n_1} \qquad S_{m_2} - S_{n_2}$

$$S_{m_1} - S_{n_1} = X_{n_1+1} + X_{n_1+2} + \dots + X_{m_1} \qquad X_{n_2+1} + X_{n_2+2} + \dots + X_{m_2} = S_{m_2} - S_{n_2}$$

the X_i are independent

$m_1 < n_2 + 1$

these X_i are all distinct,

So these sums are independent.

can take any value in $\mathbb{R}$

The Wiener Process

The Wiener process is a **continuous time** version of the random walk.

Definition 14.12: Wiener Process

defined for all times $t \geq 0$

A continuous-time, continuous-space stochastic process (W_t) is a Wiener process if it satisfies the following conditions:

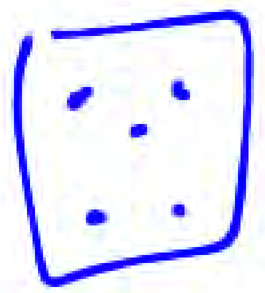
1. $W_0 = 0$;
2. For any two **disjoint** intervals $(a_1, a_2]$, $(b_1, b_2]$ the random variables $W_{a_2} - W_{a_1}$ and $W_{b_2} - W_{b_1}$ are independent;
3. If $s, t \geq 0$ then $(W_{t+s} - W_t) \sim N(0, s)$;
4. With probability 1 the map $t \mapsto W_t$ is continuous.

"Same" independence property as simple random walk.

Variance s

The Wiener process is also often called **standard Brownian motion**.

A realisation of a dice is



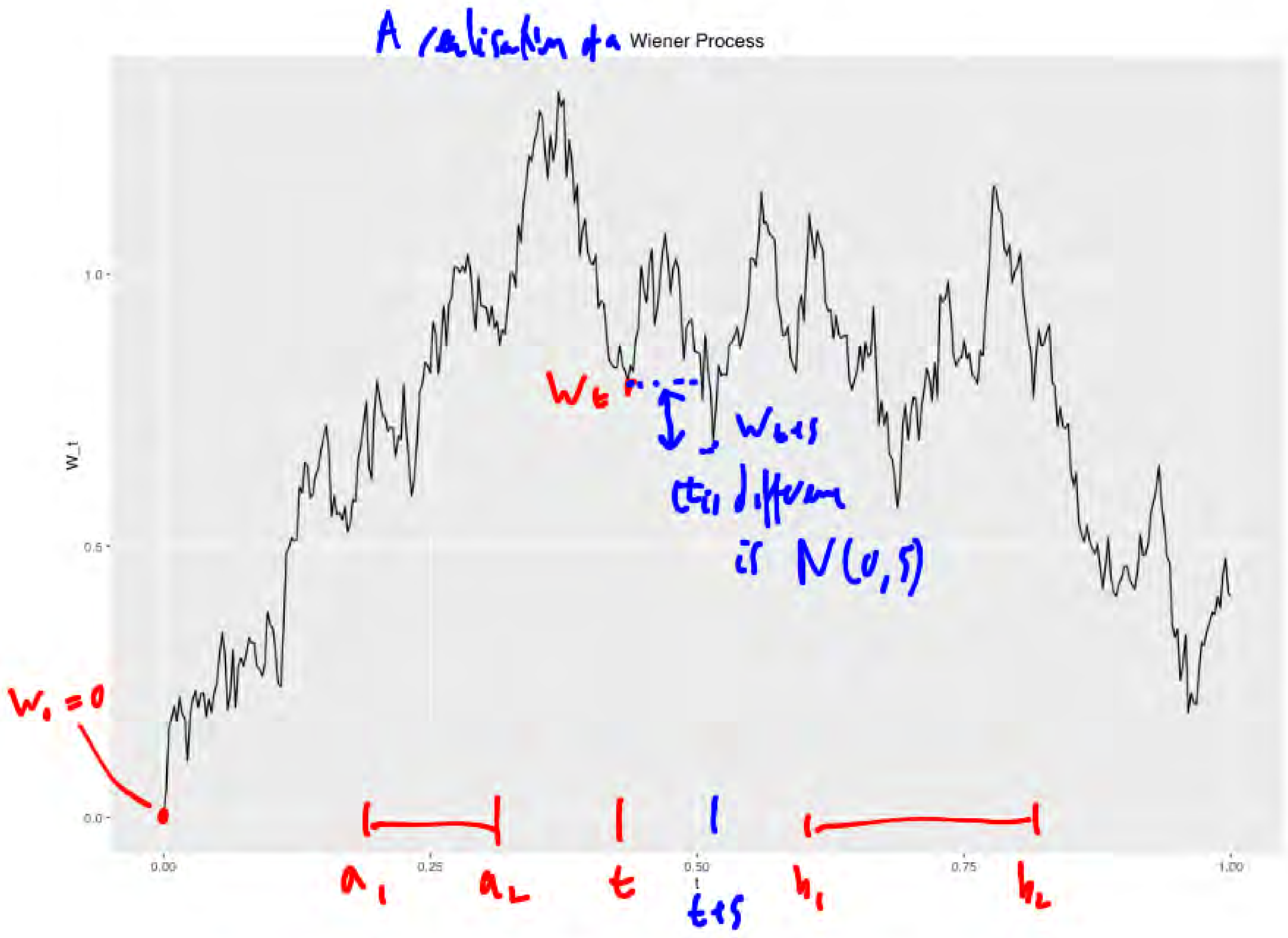


Figure: A realisation of the Wiener Process

realisation are sometimes called Brownian motion.

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2. For any two **disjoint** intervals $(a_1, a_2]$, $(b_1, b_2]$ the random variables $W_{a_2} - W_{a_1}$ and $W_{b_2} - W_{b_1}$ are independent;
3. If $s, t \geq 0$ then $(W_{t+s} - W_t) \sim N(0, s)$;
4. With probability 1 the map $t \mapsto W_t$ is continuous.

$W_t = W_t - 0 = W_t - W_0 \sim N(0,t)$

In particular for all times $t \geq 0$ we have $W_t \sim N(0, t)$ so

- $\mathrm{E}(W_t) = 0$ ✓
- $\mathrm{Var}(W_t) = t$. ✓

A Wiener process is a martingale

$E(W_t) = 0$

Lemma 14.13: Wiener process is a martingale

Premise

Let (W_t) be a Wiener process and fix times $s \leq t$, then

Conclusion

$$\underbrace{\mathrm{E}\left(W_t \mid \overbrace{\{W_\tau : \tau \leq s\}}^{\text{all the previous values up to time } s}\right)}_{\text{conditional martingale}} = W_s.$$

So increments are independent.

$W_t - W_s$ independent from $W_s - W_0$

Proof.

Essentially this follows from the fact that for $t \geq s$ the intervals $(s, t]$ and $(0, s]$ are disjoint, so the random variables $W_t - W_s$ and $W_s - W_0$ are independent. Consequently,

Linearity Theorem 10.45.

$$
\begin{aligned}
\mathrm{E}\left(W_t \mid W_s - W_0\right) - \mathrm{E}\left(W_s \mid W_s - W_0\right) &= \mathrm{E}\left(W_t - W_s \mid W_s - W_0\right) \\
&= \mathrm{E}\left(W_t - W_s\right) \\
&= \mathrm{E}\left(W_t\right) - \mathrm{E}\left(W_s\right) = 0
\end{aligned}
$$

as $W_t - W_s$ and $W_s - W_0$ are independent Theorem 10.45.

Linearity of expected value.

Proof.

Essentially this follows from the fact that for $t \geq s$ the intervals $(s, t]$ and $(0, s]$ are disjoint, so the random variables $W_t - W_s$ and $W_s - W_0$ are independent. Consequently,

$$\begin{aligned} \mathrm{E}\left(W_t \mid W_s - W_0\right) - \mathrm{E}\left(W_s \mid W_s - W_0\right) &= \mathrm{E}\left(W_t - W_s \mid W_s - W_0\right) \\ &= \mathrm{E}\left(W_t - W_s\right) \\ &= \mathrm{E}\left(W_t\right) - \mathrm{E}\left(W_s\right) = 0 \end{aligned}$$

hence

Theorem 10.45

$$\begin{aligned} \mathrm{E}\left(W_t \mid W_s - W_0\right) - \mathrm{E}\left(W_s \mid W_s\right) &= 0 \\ \mathrm{E}\left(W_t \mid W_s - W_0\right) - W_s &= 0 \\ \mathrm{E}\left(W_t \mid W_s - W_0\right) &= W_s. \end{aligned}$$

$\square$

Important properties

Lemma 14.14: Properties of Wiener realisations

Premise

Suppose that $w(t)$ is a realisation of a Wiener process

Conclusion

- $w(t)$ is continuous (almost surely)
- $w'(t)$ does not exist for almost every t (almost surely)
- $w(t)$ satisfies the law of the iterated logarithm:

$$\limsup_{t\to\infty} \frac{|w(t)|}{\sqrt{2t\log\log t}} = 1, \qquad \text{(almost surely)}$$

$|w(t)| \leq \sqrt{2t \log \log t}$ for large enough t.

$\sqrt{2t\log\log t}$

Figure: A realisation of the Wiener Process compare to simple random walk.

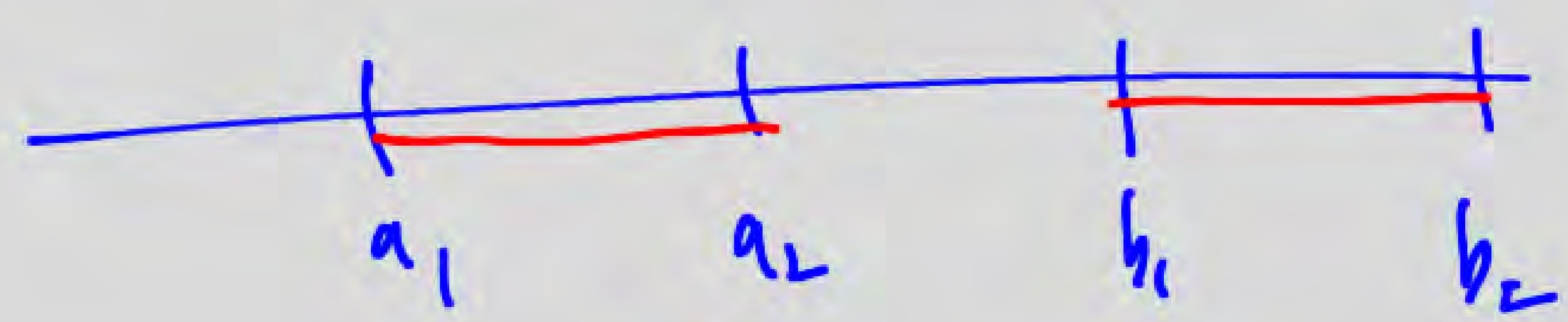
- for any $\tau > 0$ the curve
$$v(t) := w(\tau + t) - w(\tau)$$
is also a realisation of the Wiener process – this is the *translation property*
- for any $c > 0$ the curve
$$v(t) := \frac{1}{\sqrt{c}} w(ct)$$
is also a realisation of the Wiener process — this is the *scaling property*;
- the curve
$$v(t) := t w\left(\frac{1}{t}\right)$$
is also a realisation of the Wiener process — this is the *time inversion property*.

$W(t)$ $\qquad c > 0$

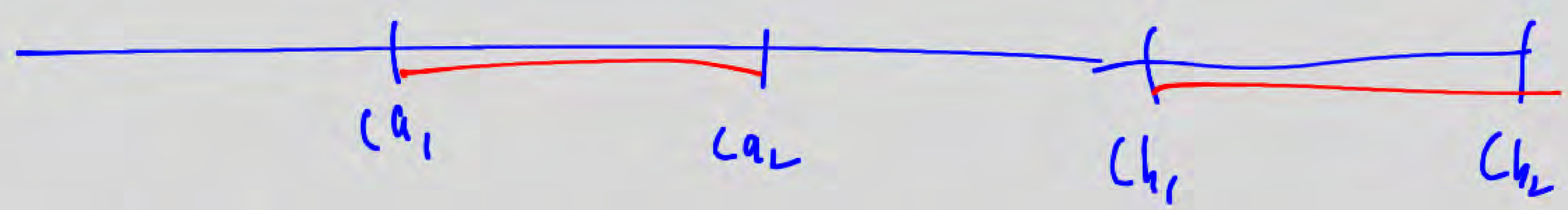
$V(t) := \frac{1}{\sqrt{c}} W(ct)$

1. $V(0) = \frac{1}{\sqrt{c}} W(c \times 0) = \frac{1}{\sqrt{c}} W(0) = 0$

2. Let $(a_1, a_2]$ and $(b_1, b_2]$ be disjoint intervals



these intervals are disjoint



$W_{ca_2} - W_{ca_1}$ is independent from $W_{cb_2} - W_{cb_1}$

So $\frac{1}{\sqrt{c}}(W_{ca_2} - W_{ca_1})$ " " " $\frac{1}{\sqrt{c}}(W_{cb_2} - W_{cb_1})$

$W_{ca_2} - W_{ca_1}$ is independent from $W_{ck_2} - W_{ck_1}$

So $\frac{1}{\sqrt{c}}\left(W_{ca_2} - W_{ca_1}\right), \dots, \frac{1}{\sqrt{c}}\left(W_{ck_2} - W_{ck_1}\right)$

$$\frac{1}{\sqrt{c}} W_{ca_2} - \frac{1}{\sqrt{c}} W_{ca_1}, \dots, \frac{1}{\sqrt{c}} W_{ck_2} - \frac{1}{\sqrt{c}} W_{ck_1}$$

$$V(a_2) - V(a_1), \dots, V(k_2) - V(k_1)$$

3. fix $t \geq 0$ $\quad V(t+s) - V(t) = \frac{1}{\sqrt{c}} W(c(t+s)) - \frac{1}{\sqrt{c}} W(ct)$

$$= \frac{1}{\sqrt{c}}\Big(W(ct+cs) - W(ct) \Big)$$

$$W(ct+cs) - W(ct) \sim N(0, cs)$$

So $\frac{1}{\sqrt{c}}\Big(W(ct+cs) - W(ct) \Big) \sim N\left(0, \left(\frac{1}{\sqrt{c}}\right)^2 cs\right) = N(0, s)$

Ex. $V(t) = \frac{1}{\sqrt{c}} W(ct)$

is continuous if and only if $W(ct)$ is continuous

if and only if $W(t)$ is continuous,

which happens with probability 1.

The reflection principle

When modelling data using the Wiener process it is often useful to know at what time we are likely to have reached a particular value. We let

Set α to be some value.

$$\boxed{T_\alpha = \inf\{t \;:\; W_t = \alpha\}}$$

smallest time where $W_t = \alpha$

so that T_α is the **first time** at which the Wiener process is equal to a given value α. For a fixed α the random variable T_α is called the *hitting time*.

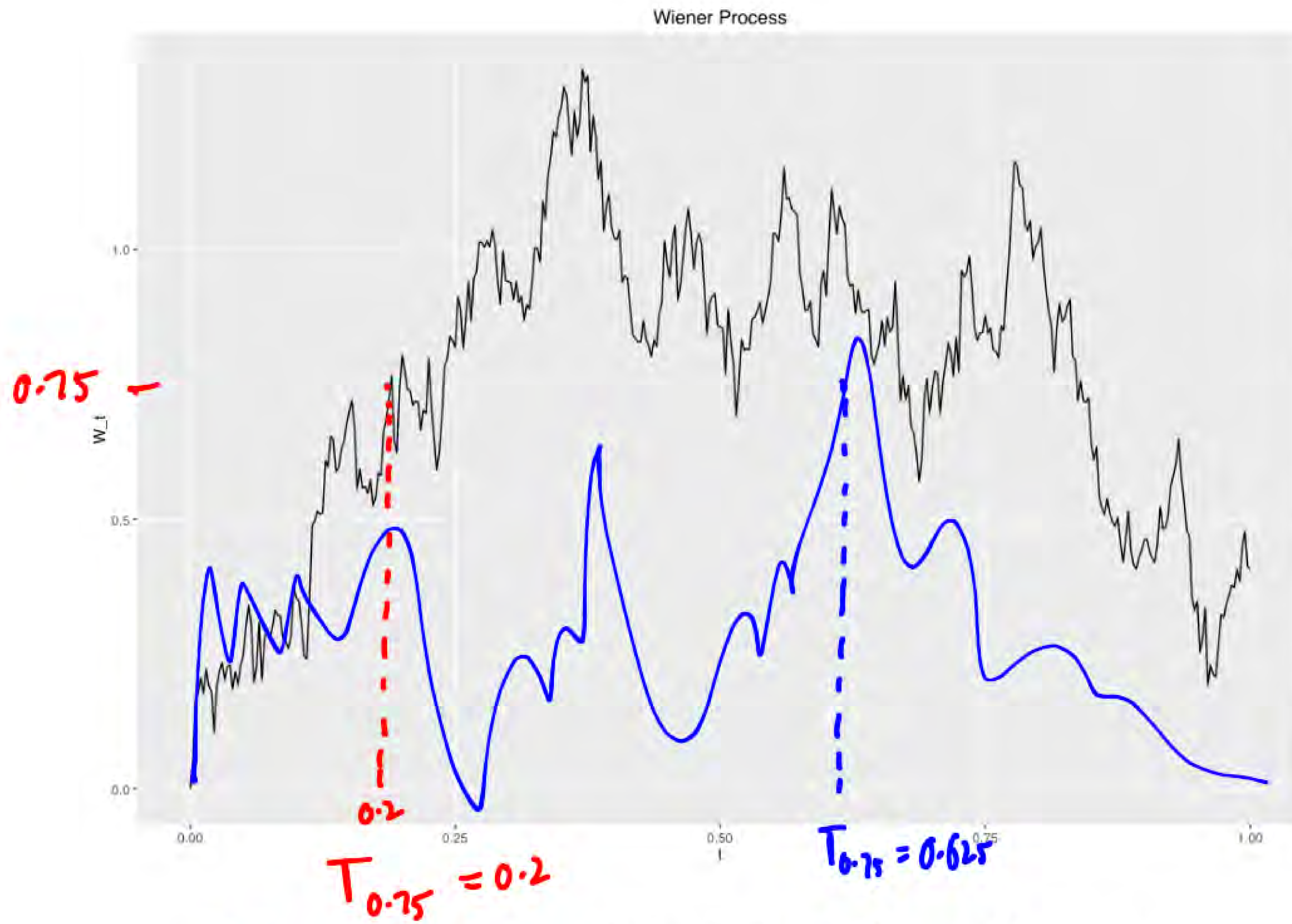


Figure: A realisation of the Wiener Process

The reflection principle is a heuristic argument that

CDF of T_α $\longrightarrow$ $P\left(T_\alpha \leq t\right) = 2P\left(W_t \geq \alpha\right)$

and proceeds as follows:
For each realisation $w(t)$ of the Wiener Process with $T_\alpha \leq t$ the 'reflected realisation'

$$v(t) := \begin{cases} w(t) & t < T_\alpha \\ 2\alpha - w(t) & t \geq T_\alpha \end{cases}$$

$V(T_\alpha) = 2\alpha - W(T_\alpha)$

$= 2\alpha - \alpha = \alpha = W(T_\alpha)$

is also a realisation of the Wiener Process with $T_\alpha \leq t$.

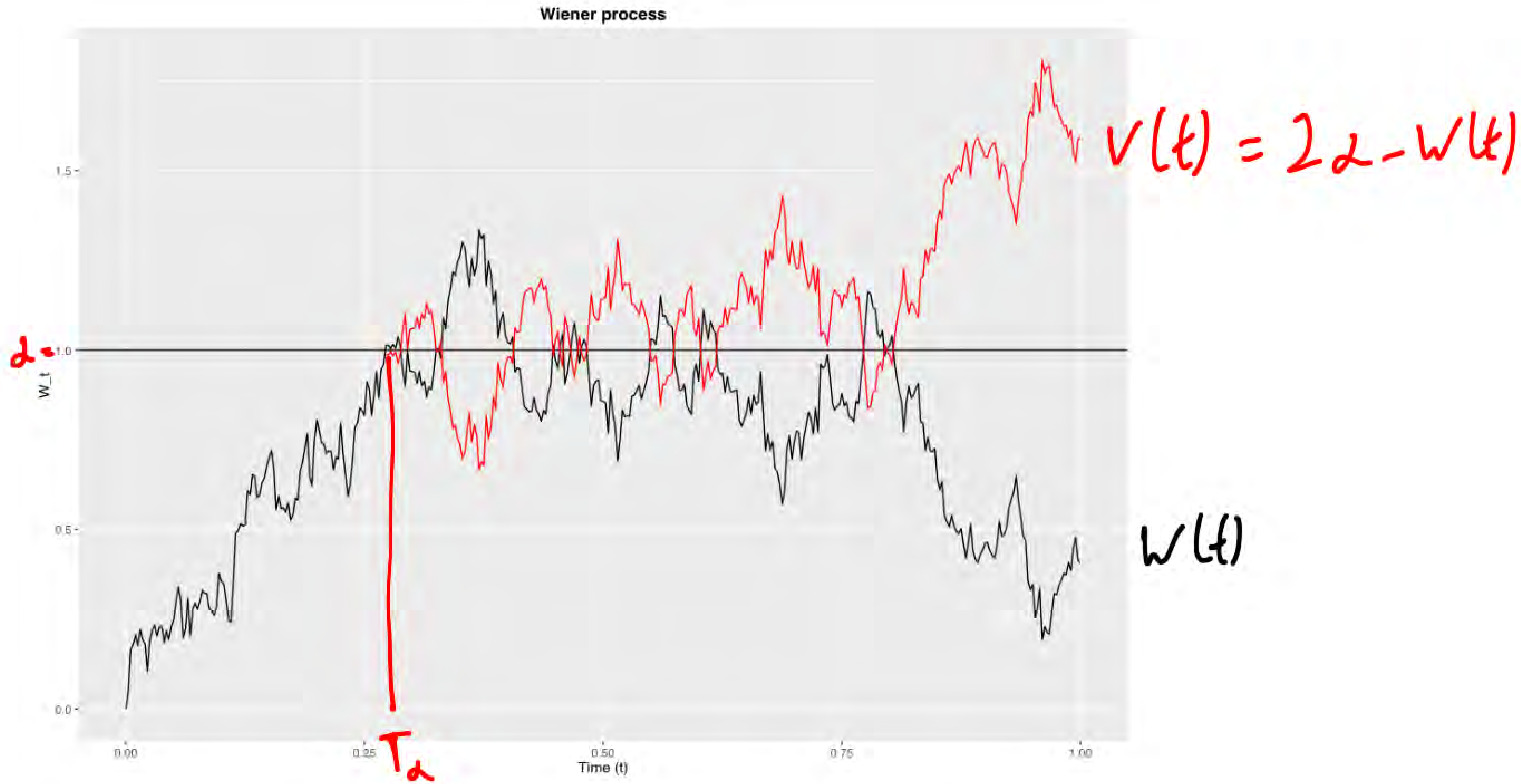


Figure: The reflection principle for the Wiener process

That is, the realisations satisfying $T_\alpha \leq t$ come in **pairs**: one of this pair has $w(t) \geq \alpha$, the other has $w(t) \leq \alpha$.

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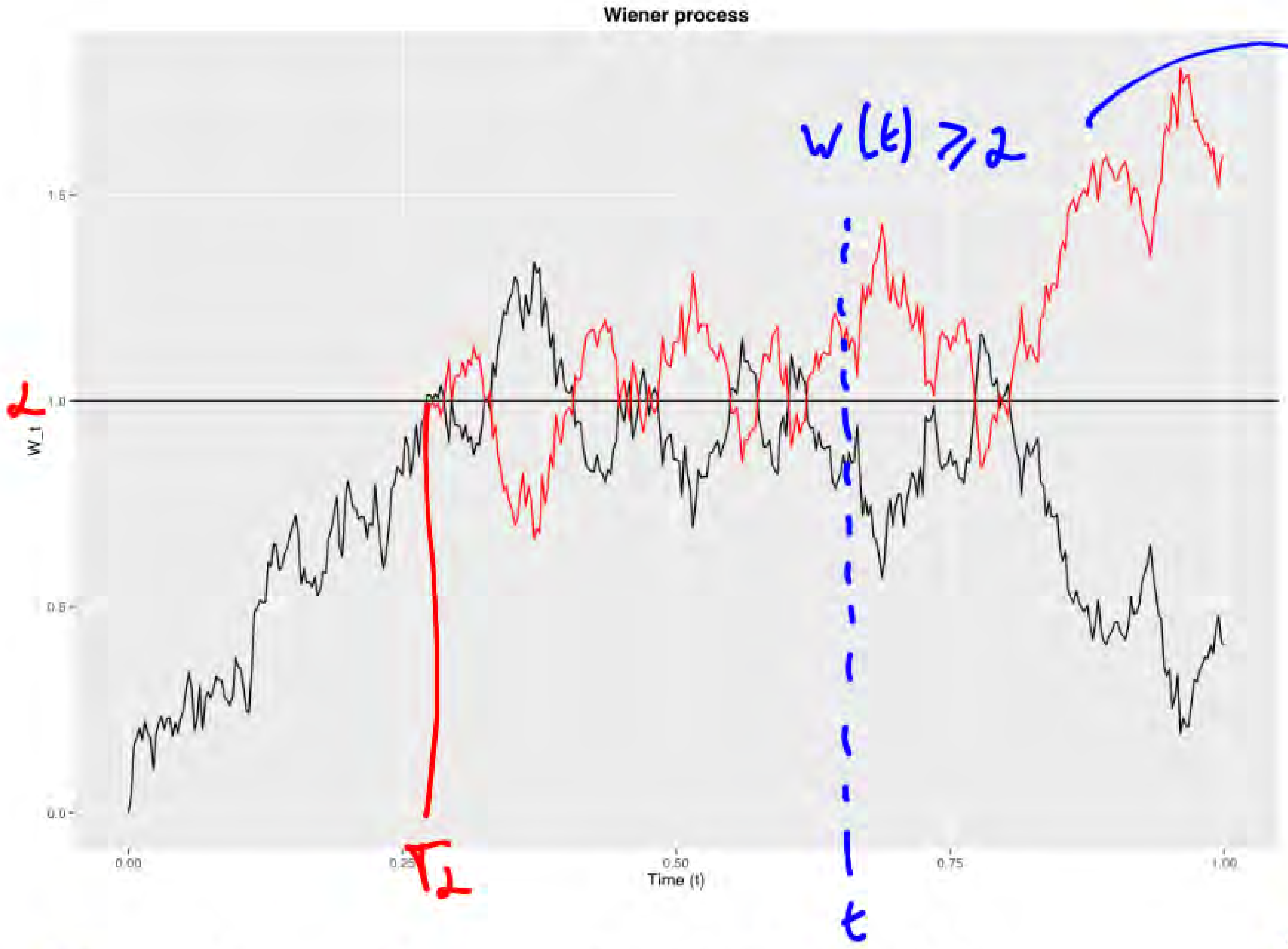
So it has hit the value α at some point before t

$t \geqslant T_\alpha$

So, if we look at all of the realisations with $w(t) \geq \alpha$ (so certainly these realisations have $T_\alpha \leq t$) it follows that this represents **half** of the realisations with $T_\alpha \leq t$. Hence,

$$P\left(T_\alpha \leq t\right) = 2P\left(W_t \geq \alpha\right)$$



for every realization satisfying this

there is another that hit α but now has $W(t) < \alpha$

Figure: The reflection principle for the Wiener process $P(T_\alpha \leq t) = 2\,P(W_t \geq \alpha)$

(More) rigorous proof

Lemma 14.15: Reflection principle

Premise

Let (W_t) be a Wiener Process and fix $\alpha > 0$, then

Conclusion

$$P\left(T_\alpha \leq t\right) = 2P\left(W_t \geq \alpha\right)$$

$$P(A) = P(A \mid E)\,P(E) + P(A \mid E^c)\,P(E^c)$$

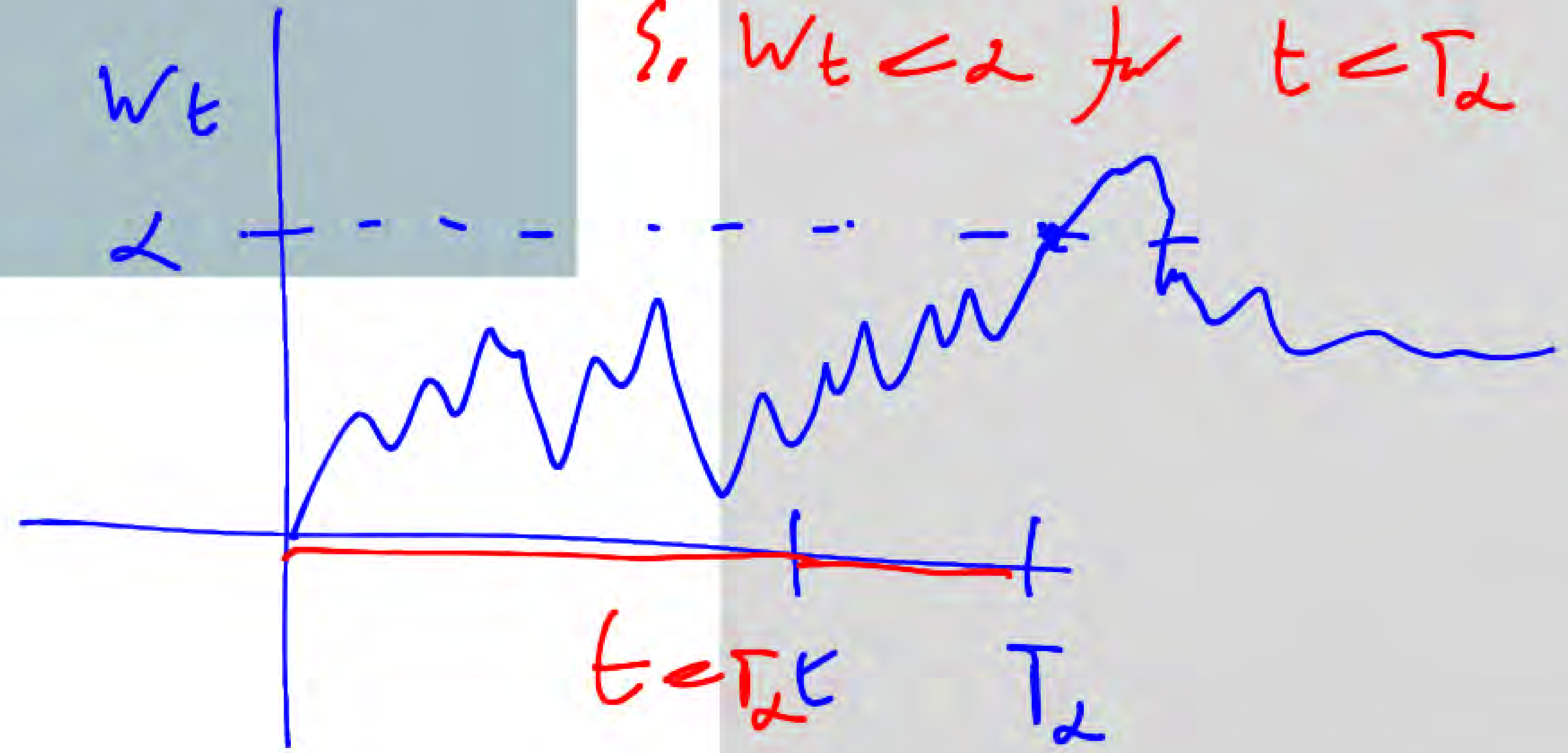
Proof:

Fix $\alpha > 0$. By the Law of Total Probability

$$P(W_t \geq \alpha) = P(W_t \geq \alpha \mid T_\alpha \leq t)\, P(\overbrace{T_\alpha \leq t}^{\text{time } t \text{ is bigger than the first time we hit } \alpha}) + \underbrace{\cancel{P(W_t \geq \alpha \mid T_\alpha > t)}}_{=0}\, P(\overbrace{T_\alpha > t}^{\text{we have not yet hit } \alpha}).$$

However, from the definition of T_α we cannot have both $W_t \geq \alpha$ and $t < T_\alpha$, hence $P(W_t \geq \alpha \mid T_\alpha > t) = 0$.

So, $W_t < \alpha$ for $t < T_\alpha$



Proof:

Fix $\alpha > 0$. By the Law of Total Probability

$$P\left(W_t \geq \alpha\right) = P\left(W_t \geq \alpha \mid T_\alpha \leq t\right) P\left(T_\alpha \leq t\right) + P\left(W_t \geq \alpha \mid T_\alpha > t\right) P\left(T_\alpha > t\right).$$

However, from the definition of T_α we cannot have both $W_t \geq \alpha$ and $t < T_\alpha$, hence $P\left(W_t \geq \alpha \mid T_\alpha > t\right) = 0$.
Consequently,

$$P\left(W_t \geq \alpha\right) = \underbrace{P\left(W_t \geq \alpha \mid T_\alpha \leq t\right)}_{= 1/2} P\left(T_\alpha \leq t\right)$$

we will show

$t = T_\alpha + (t - T_\alpha)$

translated Wiener process realization (is also a realization of the Wiener process.

Next, observe that

$$
\begin{aligned}
P\left(W_t \geq \alpha \mid T_\alpha \leq t\right) &= P\left(W_{T_\alpha+(t-T_\alpha)} \geq \alpha \mid T_\alpha \leq t\right) \\
&= P\left(W_{T_\alpha+(t-T_\alpha)} - \alpha \geq 0 \mid T_\alpha \leq t\right) \\
&= P\left(V_{t-T_\alpha} \geq 0 \mid T_\alpha \leq t\right)
\end{aligned}
$$

$= P(V_{t-T_\alpha} \geq 0)$ as it is independent of $T_\alpha \leq t$

where V_t is the Wiener process defined by $V_t = W_{T_\alpha+t} - W_{T_\alpha}$, which is independent of T_α. Hence

$V_t = W_{T_\alpha+t} - \alpha$ $\quad$ $V_{t-T_\alpha} = W_{T_\alpha+t-T_\alpha} - \alpha$

$$
P\left(W_t \geq \alpha \mid T_\alpha \leq t\right) = P\left(V_{t-T_\alpha} \geq 0\right) = \frac{1}{2}
$$

and the result follows.

$V_{t-T_\alpha} \sim N(0, t-T_\alpha)$

Corollary 14.16: Distribution of T_α

Premise

Let (W_t) be a Wiener Process and fix $\alpha \in \mathbb{R}$, then

Conclusion

The ~~cdf~~ CDF of T_α is

$$F_{T_\alpha}(t) = P(T_\alpha \leq t) = \frac{2}{\sqrt{t}\sqrt{2\pi}} \int_\alpha^\infty e^{-x^2/2t} \, dx$$

$$\frac{d}{dt} \int_\alpha^\infty e^{-x^2/2t} \, dx$$

and the ~~pdf~~ PDF of T_α is

$$f_{T_\alpha}(t) = \frac{\alpha}{t^{\frac{3}{2}}} \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2t}.$$

$$\int_\alpha^\infty \frac{d}{dt} e^{-x^2/2t}$$

Proof.

From the previous lemma:

$W_0 = 0$

$$
\begin{aligned}
P(T_\alpha \le t) &= 2P(W_t \ge \alpha) \\
&= 2P(W_t - W_0 \ge \alpha) \\
&= 2\int_\alpha^\infty \frac{1}{\sqrt{t}\sqrt{2\pi}} e^{-x^2/2t}\, dx
\end{aligned}
$$

Normal dist $N(0,t)$

PDF of $N(0,t)$

as $W_t - W_0 \sim N(0,t)$

$$
= \frac{2}{\sqrt{t}\sqrt{2\pi}} \int_\alpha^\infty e^{-x^2/2t}\, dx.
$$

$\square$

If we change variable $y = \frac{\alpha^2 t}{x^2}$

$$
= \frac{2}{\sqrt{2\pi}} \int_0^t e^{-\alpha^2/2y}\, y^{-3/2}\, dy
$$

Wiener process with drift

The Wiener Process allows us to include continuous statistical noise into more general models. The simplest is the Wiener Process with drift, which is based on a simple linear function.

Definition 14.17: Wiener process with drift

A Wiener process with drift μ and variance σ^2 is defined by the relation

$$\boxed{B_t = \sigma W_t + \mu t.}$$

where W_t is a Wiener Process.

Note here that

$$E(B_t) = \mu t, \qquad Var(B_t) = \sigma^2 t.$$

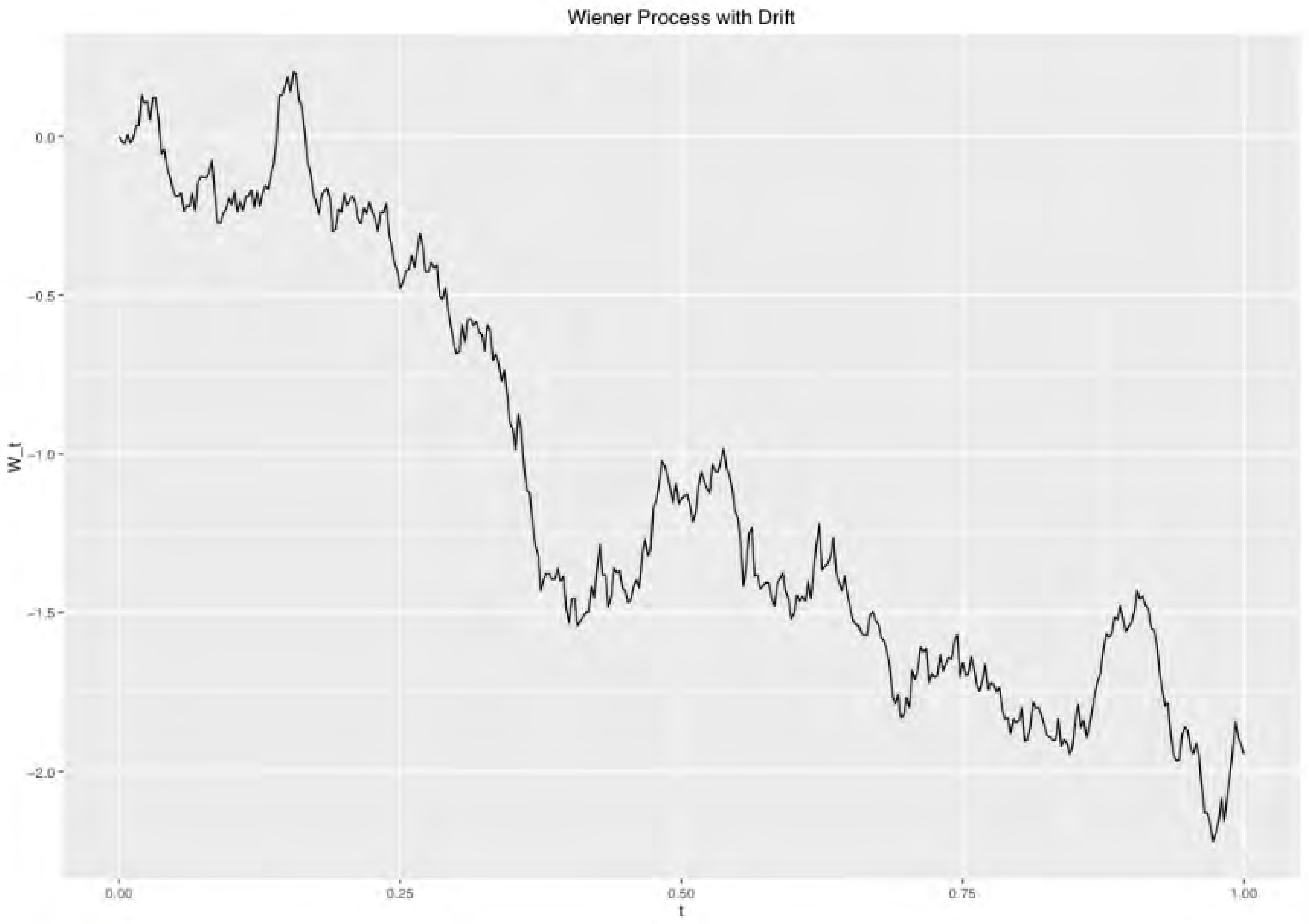


Figure: Wiener process with drift $\mu = 1$ and $\sigma = 0.5$

The hitting time for the Wiener process with drift if defined in a similar way to the Wiener process:

$$T_\alpha = \inf\{t \colon B_t = \alpha\}.$$

Using the reflection principle it is possible to show that T_α has pdf

$$f_{T_\alpha}(t) = \frac{|\alpha - \mu t|}{\sigma\sqrt{2\pi t^3}} e^{-(\alpha-\mu t)^2/(2\sigma^2 t)}, \qquad t \geq 0. \tag{1}$$

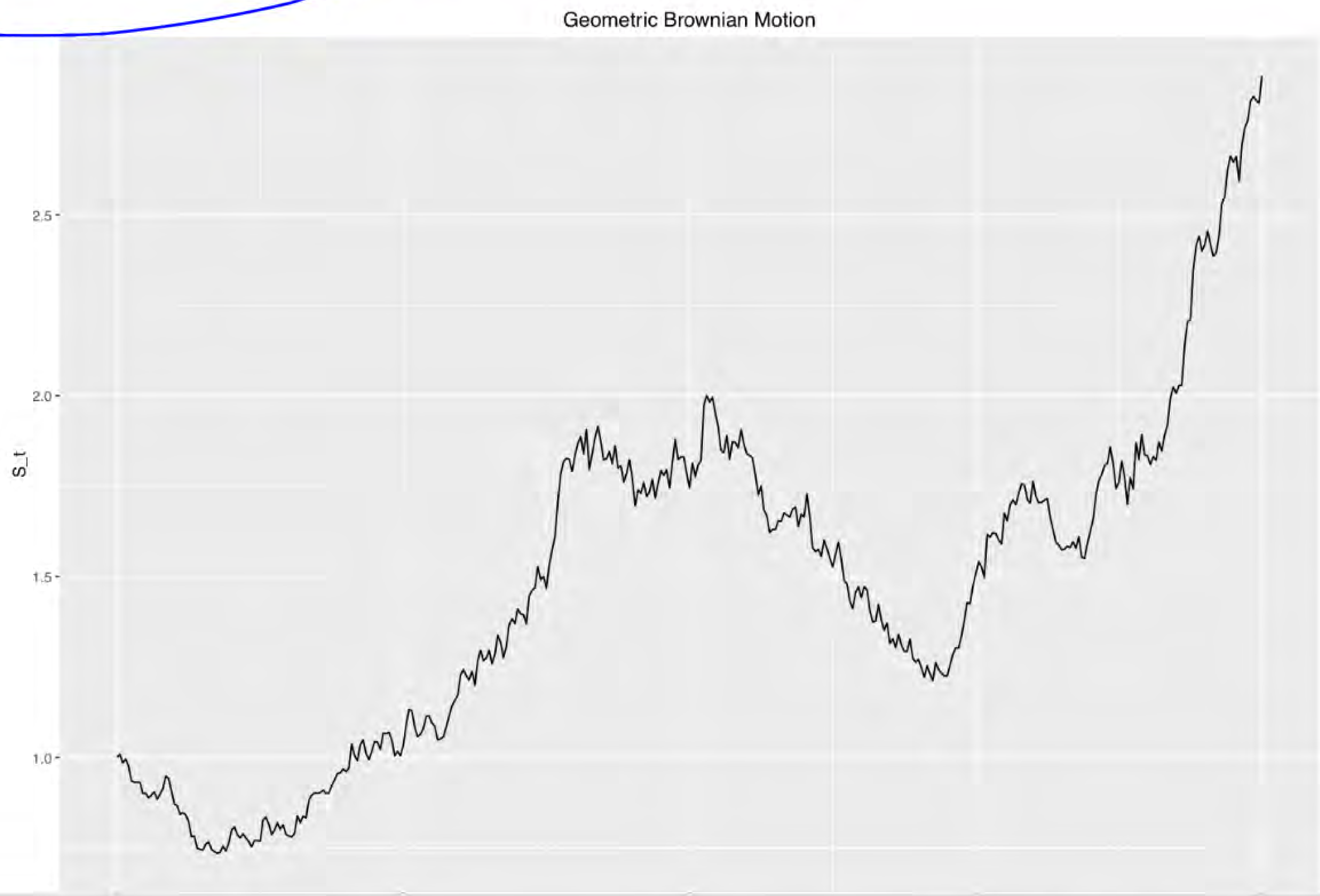
An interesting consequence is that for $\alpha > 0$

probability that we ever hit α.

$$P\left(T_\alpha < \infty\right) = \begin{cases} 1 & \mu \geq 0 \\ e^{2\alpha\mu} & \mu < 0 \end{cases}$$

(which we can check by integrating the pdf (1)).
This means that for $\alpha > 0$ and $\mu \geq 0$ then B_t will almost surely hit α at some point in the future. However, if $\mu < 0$ then this occurs with probability less than 1.

Geometric Brownian motion



The Wiener process with drift produces both positive and negative values and so does not suitable model things like stock prices in financial mathematics.
Instead a model that has become popular, largely because of its appearance in the Black-Scholes equation, is geometric Brownian motion.

Definition 14.18: Geometric Brownian Motion

Suppose that W_t is a Wiener process then Geometric Brownian Motion is the process S_t defined by

$$\boxed{S_t = S_0 e^{\sigma W_t + \mu t}}$$

where $S_0 > 0$ is the initial value.

It is possible to find the expected value of geometric Brownian motion using the moment generating function, $M_X(y) = \mathrm{E}(\mathrm{e}^{yX})$. MGF of X

$$
\begin{aligned}
\mathrm{E}(S_t) &= \mathrm{E}(S_0 \mathrm{e}^{\sigma W_t + \mu t}) \\
&= \mathrm{E}(S_0 \mathrm{e}^{\mu t} e^{\sigma W_t}) \\
&= S_0 \mathrm{e}^{\mu t}\, \mathrm{E}(e^{\sigma W_t}) \quad \text{MGF of } W_t \text{ evaluated at } \sigma \\
&= S_0 \mathrm{e}^{\mu t} M_{W_t}(\sigma).
\end{aligned}
$$

However $W_t \sim N(0, t)$ so that $M_{W_t}(\sigma) = \mathrm{e}^{\sigma^2 t/2}$

$$
\begin{aligned}
\mathrm{E}(S_t) &= S_0 \mathrm{e}^{\mu t} e^{\sigma^2 t/2} \\
&= S_0 \mathrm{e}^{(\mu + \sigma^2/2)t}.
\end{aligned}
$$

We see that the random variable S_t has constant expected value iff $\mu = -\sigma^2/2$.

t	b
t_1	$b(t_1)$
t_2	$b(t_2)$
t_3	$b(t_3)$
$\vdots$	$\vdots$

Modelling data

Suppose we have data $b(t_1)$, $b(t_2)$, $b(t_3)$, ... at evenly spaced times (i.e. $t_{k+1} = t_k + h$ for some h).
If this data can be modelled by a Wiener Process with drift then the increments $b(t_{k+1}) - b(t_k)$ should be distributed like

we used δ in our code

$$
\begin{aligned}
\underbrace{b(t_{k+1}) - b(t_k)}_{\text{data}} \sim \underbrace{B(t_{k+1}) - B(t_k)}_{\text{process}} &= B(t_k + h) - B(t_k) \\
&= \sigma W(t_k + h) + \mu(t_k + h) - \sigma W(t_k) - \mu(t_k) \\
&= \sigma (W(t_k + h) - W(t_k)) + \mu h \\
&\sim \sigma N(0, h) + \mu h \\
&\sim N(\mu h, \sigma^2 h)
\end{aligned}
$$

h is the time-step.

We can therefore derive estimates for μ and σ using the usual techniques from Chapter 13.

parameter that we need to find.

In many situations we only have have **one** realisation of the process to investigate we are unable to check that, for example, the random variable $W_{1+2} - W_1$ is distributed according to $N(0, 2)$ - we only have one data point from this distribution!
To check the suitability of the Wiener process with drift we can check that

- for each n the increments $b(t + nh) - b(t)$ do not depend on the value of t, then
- for each n, the increments $b(t + nh) - b(t)$ have the appropriate normal distribution (the data is obtained by taking all the different values of t), and

If we have many realisations, then we can investigate each random variable $B_(t + nh) - B(t)$ separately to verify that the hypotheses are satisfied.

So we can assume that the increments come from the same distribution. Is this a normal distribution? We can now check as we have lots of data.

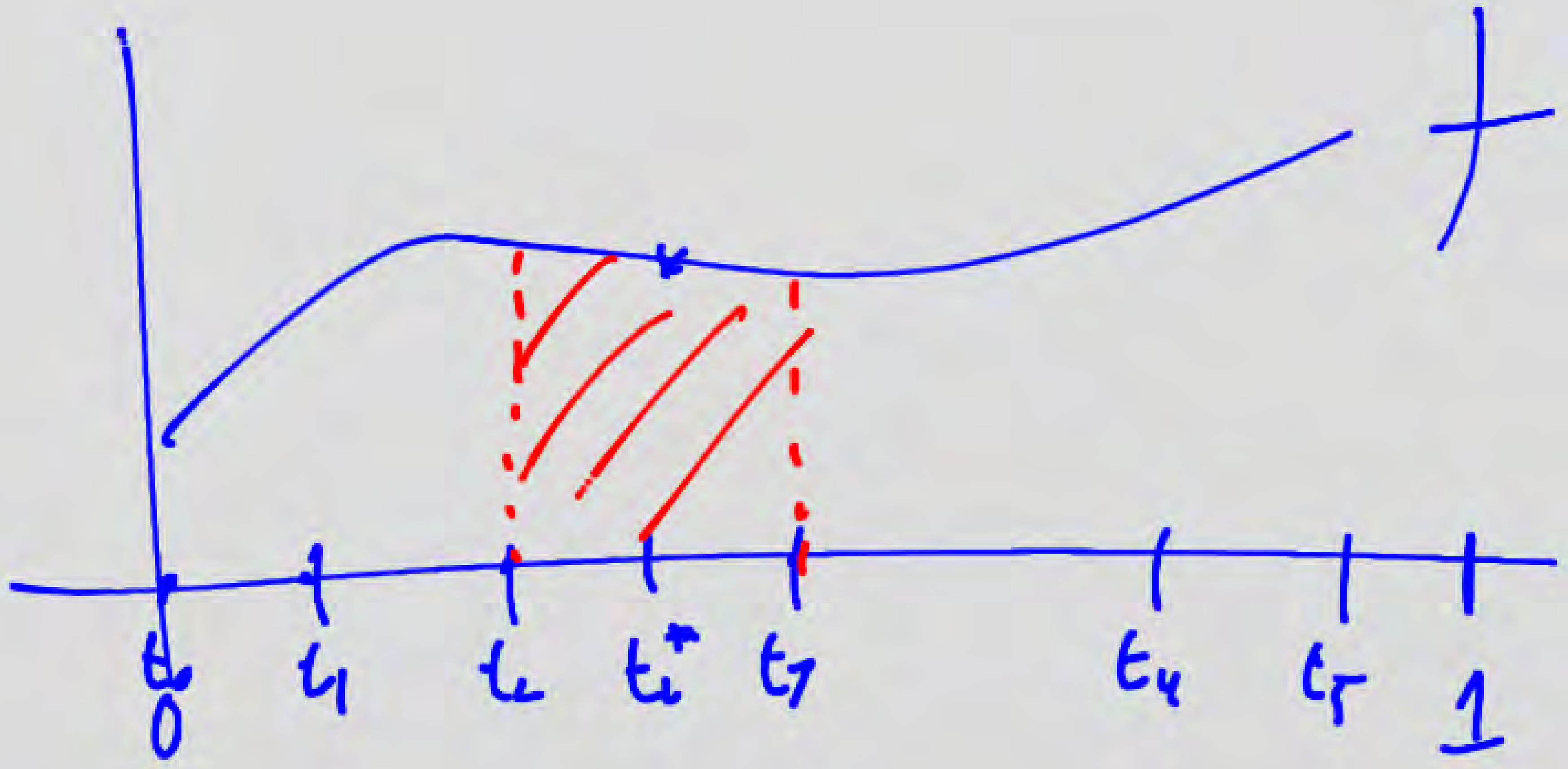
Modelling data with Geometric Brownian Motion

To fit a geometric Brownian motion model to data $s(t_k)$ we observe that $s(t_k)$ can be modelled by $S_t = S_0 e^{\sigma W_t + \mu t}$ iff $\log s(t_k)$ can be modelled by a Wiener Process with Drift, so we apply the techniques from the previous section.

$$\log(S_t) = \log(S_0 e^{\sigma W_t + \mu t})$$

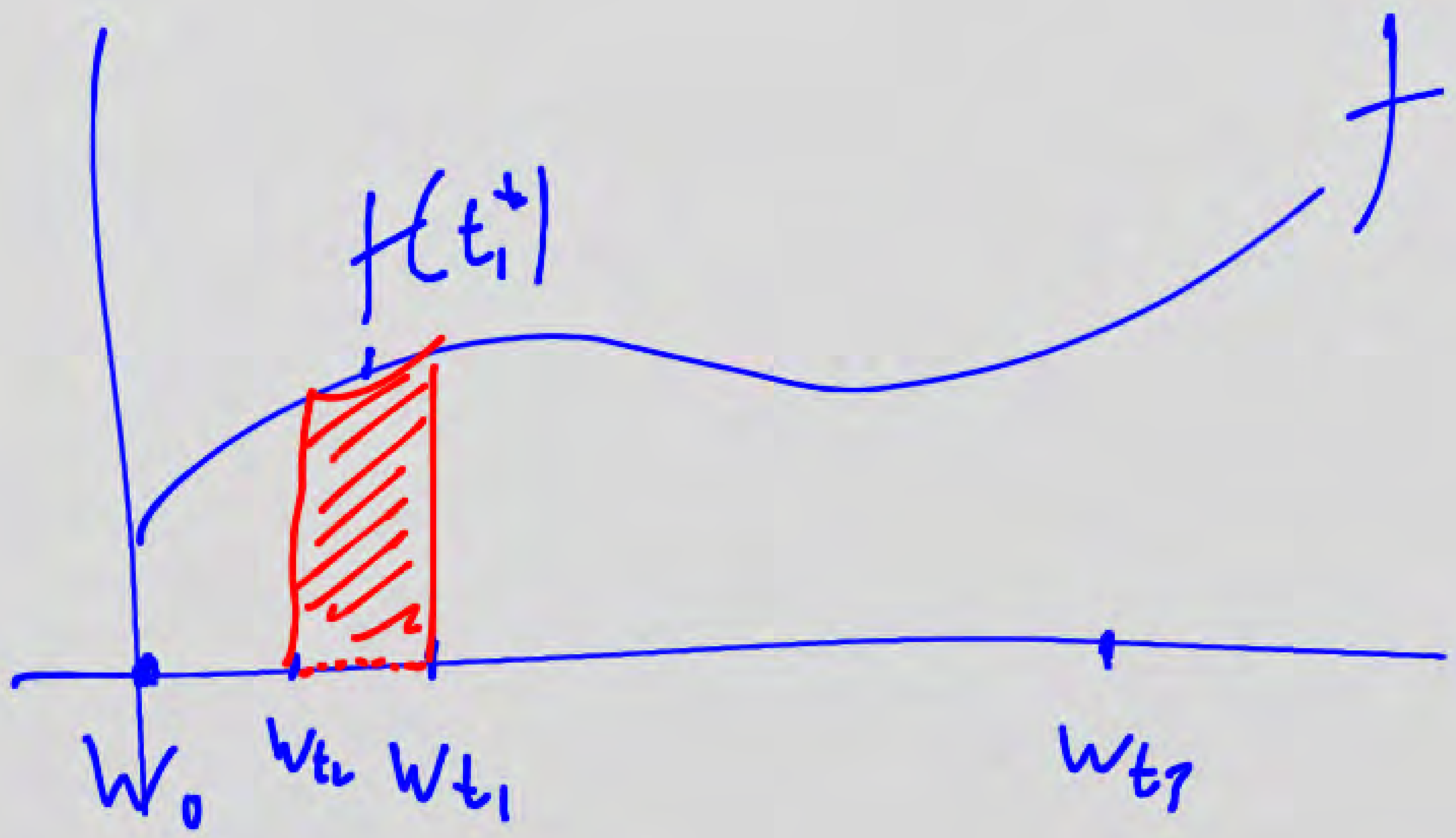
$$= \log(S_0) + \sigma W_t + \mu t$$

Take log and decide if result comes from Brownian Motion with drift.



$$\int f\,dx$$

 $= (t_3 - t_2)\, t_2^*$



$$\int f\,dW_t$$

 $f(t_1^*)\,(W_{t_2} - W_{t_1})$

Stochastic Differential Equations

X is a Wiener process $dX_t = dW_t$ increments of X are increments of Wiener process.

X is a Wiener process with drift $dX_t = \mu dt + \sigma dW_t$ ← Wiener process with drift

GBM: $dX_t = \mu X_t + \sigma X_t dW_t$

increments depend on current value.

Ornstein-Uhlenbeck process $dX_t = \theta(\mu - X_t)dt + \sigma dW_t$

Mean reverting

Modelling e.g. Currency Forex.