

# DEGREE BOUNDS FOR MODULAR COVARIANTS

JONATHAN ELMER AND MÜFIT SEZER

ABSTRACT. Let  $V, W$  be representations of a cyclic group  $G$  of prime order  $p$  over a field  $\mathbb{k}$  of characteristic  $p$ . The module of covariants  $\mathbb{k}[V, W]^G$  is the set of  $G$ -equivariant polynomial maps  $V \rightarrow W$ , and is a module over  $\mathbb{k}[V]^G$ . We give a formula for the Noether bound  $\beta(\mathbb{k}[V, W]^G, \mathbb{k}[V]^G)$ , i.e. the minimal degree  $d$  such that  $\mathbb{k}[V, W]^G$  is generated over  $\mathbb{k}[V]^G$  by elements of degree at most  $d$ .

## 1. INTRODUCTION

Let  $G$  be a finite group,  $\mathbb{k}$  field and  $V, W$  a pair of  $\mathbb{k}G$ -modules. Then  $G$  acts on the set of polynomial functions  $V \rightarrow W$  by the formula

$$(g\phi)(v) = g\phi(g^{-1}v).$$

A covariant is a  $G$ -equivariant function  $V \rightarrow W$  and an invariant is a covariant  $V \rightarrow \mathbb{k}$ . The set of invariants is denoted as  $\mathbb{k}[V]^G$  and the set of covariants as  $\mathbb{k}[V, W]^G$ . For  $f \in \mathbb{k}[V]^G$  and  $\phi \in \mathbb{k}[V, W]^G$  we define the product

$$f\phi(v) = f(v)\phi(v).$$

Then  $\mathbb{k}[V]^G$  is a  $\mathbb{k}$ -algebra and  $\mathbb{k}[V, W]^G$  is a  $\mathbb{k}[V]^G$ -module. Modules of covariants in the non-modular case ( $|G| \neq 0 \in \mathbb{k}$ ) were studied by Chevalley [3], Sheppard-Todd [10], Eagon-Hochster [7]. In the modular case far less is known, but recent work of Broer and Chuai [1] has shed some light on the subject. A systematic attempt to construct generating sets for modules of covariants when  $G$  is a cyclic group of order  $p$  was begun by the first author in [5].

Let  $A = \bigoplus_{d \geq 0} A_d$  be any graded  $\mathbb{k}$ -algebra and  $M = \sum_{d \geq 0} M_d$  any graded  $A$ -module. Then the Noether Bound  $\beta(A)$  is defined to be the minimum degree  $d > 0$  such that  $A$  is generated by the set  $\{a : a \in A_k, k \leq d\}$ . Similarly,  $\beta(M, A)$  is defined to be the minimum degree  $d > 0$  such that  $M$  is generated over  $A$  by the set  $\{m : m \in M_k, k \leq d\}$ , and we sometimes write  $\beta(M) = \beta(M, A)$  when the context is clear.

Noether famously showed that  $\beta(\mathbb{C}[V]^G) \leq |G|$  for arbitrary finite  $G$ , but computing Noether bounds in the modular case is highly nontrivial. When  $G$  is cyclic of prime order, the second author along with Fleischmann, Shank and Woodcock [6] determined the Noether bound for any  $\mathbb{k}G$ -module. The purpose of this short article is to find results similar to those in [6] for covariants. Our main result can be stated concisely as:

**Theorem 1.** *Let  $G$  be a cyclic group of order  $p$ ,  $\mathbb{k}$  a field of characteristic  $p$ ,  $V$  a reduced  $\mathbb{k}G$ -module and  $W$  a nontrivial indecomposable  $\mathbb{k}G$ -module. Then*

$$\beta(\mathbb{k}[V, W]^G) = \beta(\mathbb{k}[V]^G)$$

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unless  $V$  is indecomposable of dimension 2.

## 2. PRELIMINARIES

For the rest of this article,  $G$  denotes a cyclic group of order  $p > 0$ , and we let  $\mathbb{k}$  be a field of characteristic  $p$ . We choose a generator  $\sigma$  for  $G$ . Over  $\mathbb{k}$ , there are  $p$  indecomposable representations  $V_1, \dots, V_p$  and each indecomposable representation  $V_i$  is afforded by a Jordan block of size  $i$ . Note that  $V_p$  is isomorphic to the free module  $\mathbb{k}G$ .

We assume that  $V$  and  $W$  are  $\mathbb{k}G$ -modules with  $W$  indecomposable and we choose a basis  $w_1, \dots, w_n$  for  $W$  so that we have

$$\sigma w_i = \sum_{1 \leq j \leq i} (-1)^{i-j} w_j,$$

for  $1 \leq i \leq n$ .

Let  $\Delta = \sigma - 1 \in \mathbb{k}G$ . We define the transfer map  $\text{Tr} : \mathbb{k}[V] \rightarrow \mathbb{k}[V]$  by  $\sum_{1 \leq i \leq p} \sigma^i$ . Notice that we also have  $\text{Tr} = \Delta^{p-1}$ .

For  $f \in \mathbb{k}[V]$  we define the *weight* of  $f$  to be the smallest positive integer  $d$  with  $\Delta^d(f) = 0$ . Note that  $\Delta^p = (\sigma - 1)^p = 0$ , so the weight of a polynomial is at most  $p$ .

A useful description of covariants is given in [5]. We include this description here for completeness.

**Proposition 2.** [5, Proposition 3] *Let  $f \in \mathbb{k}[V]$  with weight  $d \leq n$ . Then*

$$\sum_{1 \leq j \leq d} \Delta^{j-1}(f) w_j \in \mathbb{k}[V, W]^G.$$

*Conversely, if*

$$f_1 w_1 + f_2 w_2 + \dots + f_n w_n \in \mathbb{k}[V, W]^G,$$

*then there exists  $f \in \mathbb{k}[V]$  with weight  $\leq n$  such that  $f_j = \Delta^{j-1}(f)$  for  $1 \leq j \leq n$ .*

For a non-zero covariant  $h = f_1 w_1 + f_2 w_2 + \dots + f_n w_n$ , we define the *support* of  $h$  to be the largest integer  $j$  such that  $f_j \neq 0$ . We denote the support of  $h$  by  $s(h)$ . We shall say  $h$  is a *transfer covariant* if there exists a non-negative integer  $k$  and  $f \in \mathbb{k}[V]$  such that  $f_1 = \Delta^k(f)$ ,  $f_2 = \Delta^{k+1}(f)$ ,  $\dots$ ,  $f_{s(h)} = \Delta^{p-1}(f)$  for some  $f \in \mathbb{k}[V]$ .

We call a homogeneous invariant in  $\mathbb{k}[V]^G$  indecomposable if it is not in the subalgebra of  $\mathbb{k}[V]^G$  generated by invariants of strictly smaller degree. Similarly, a homogeneous covariant in  $\mathbb{k}[V, W]^G$  is indecomposable if it does not lie in the submodule of  $\mathbb{k}[V, W]^G$  generated by covariants of strictly smaller degree.

## 3. UPPER BOUNDS

We first prove a result on decomposability of a transfer covariant. In the proof below we set  $\gamma = \beta(\mathbb{k}[V], \mathbb{k}[V]^G)$ .

**Proposition 3.** *Let  $f \in \mathbb{k}[V]$  and  $h = \Delta^k(f)w_1 + \Delta^{k+1}(f)w_2 + \dots + \Delta^{p-1}(f)w_{s(h)}$  be a transfer covariant of degree  $> \gamma$ . Then  $h$  is decomposable.*

*Proof.* Let  $g_1, \dots, g_t$  be a set of homogeneous polynomials of degree at most  $\gamma$  generating  $\mathbb{k}[V]$  as a module over  $\mathbb{k}[V]^G$ . So we can write  $f = \sum_{1 \leq i \leq t} q_i g_i$ , where each  $q_i \in \mathbb{k}[V]^G_+$  is a positive degree invariant. Since  $\Delta^j$  is  $\mathbb{k}[V]^G$ -linear, we have  $\Delta^j(f) = \sum_{1 \leq i \leq t} q_i \Delta^j(g_i)$  for  $k \leq j \leq p-1$ . It follows that

$$h = \sum_{1 \leq i \leq t} q_i (\Delta^k(g_i)w_1 + \dots + \Delta^{p-1}(g_i)w_{s(h)}).$$

Note that  $\Delta^k(g_i)w_1 + \cdots + \Delta^{p-1}(g_i)w_{s(h)}$  is a covariant for each  $1 \leq i \leq t$  by Proposition 2. We also have  $q_i \in \mathbb{k}[V]_+^G$  so it follows that  $h$  is decomposable.  $\square$

Write  $V = \bigoplus_{j=1}^m V_{n_j}$ . Note that  $\mathbb{k}[V \oplus V_1, W]^G = (S(V^*) \otimes S(V_1^*)) \otimes W^G = \mathbb{k}[V, W]^G \otimes \mathbb{k}[V_1]$ . Therefore we will assume that  $n_j > 1$  for all  $j$ ; such representations are called reduced. Choose a basis  $\{x_{i,j} \mid 1 \leq i \leq n_j, 1 \leq j \leq m\}$  for  $V^*$ , with respect to which we have

$$\sigma(x_{i,j}) = \begin{cases} x_{i,j} + x_{i+1,j} & i < n_j; \\ x_{i,j} & i = n_j. \end{cases}$$

This induces a multidegree  $\mathbb{k}[V] = \bigoplus_{\mathbf{d} \in \mathbb{N}^m} \mathbb{k}[V]_{\mathbf{d}}$  which is compatible with the action of  $G$ . For  $1 \leq j \leq m$  we define  $N_j = \prod_{\sigma \in G} \sigma x_{1,j}$ , and note that the coefficient of  $x_{1,j}^p$  in  $N_j$  is 1. Given any  $f \in \mathbb{k}[V_{n_j}]$ , we can therefore perform long division, writing

$$(1) \quad f = q_j N_j + r$$

where  $q_j \in \mathbb{k}[V_{n_j}]$  for all  $j$  and  $r \in \mathbb{k}[V_{n_j}]$  has degree  $< p$  in the variable  $x_{1,j}$ . This induces a vector space decomposition

$$\mathbb{k}[V_{n_j}] = N_j \mathbb{k}[V_{n_j}] \oplus B_j$$

where  $B_j$  is the subspace of  $\mathbb{k}[V_{n_j}]$  spanned by monomials with  $x_{1,j}$ -degree  $< p$ , but the form of the action implies that  $B$  and its complement are  $\mathbb{k}G$ -modules, so we obtain a  $\mathbb{k}G$ -module decomposition. Since  $\mathbb{k}[V] = \bigotimes_{j=1}^m \mathbb{k}[V_{n_j}]$ , it follows that

$$\mathbb{k}[V] = N_j \mathbb{k}[V] \oplus (B_j \otimes \mathbb{k}[V']),$$

where  $V' = V_{n_1} \oplus \cdots \oplus V_{n_{j-1}} \oplus V_{n_{j+1}} \cdots \oplus V_{n_m}$ . From this decomposition it follows that if  $M$  is a  $\mathbb{k}G$  direct summand of  $\mathbb{k}[V]_d$ , then  $N_j M$  is a  $\mathbb{k}G$  direct summand of  $\mathbb{k}[V]_{d+p}$  with the same isomorphism type. Further, any  $f \in \mathbb{k}[V]^G$  can be written as  $f = qN_j + r$  with  $q \in \mathbb{k}[V]^G$  and  $r \in (B_j \otimes \mathbb{k}[V'])^G$ . If in addition  $\deg(f) = (d_1, d_2, \dots, d_m)$  with  $d_j > p - n_j$ , then the degree  $d_j$  homogeneous component of  $B_j$  is free by [8, 2.10] and since tensoring a module with a free module gives a free module we may further assume that  $r$  is in the image of the transfer map.

If  $h = \sum_{i=1}^{s(h)} \Delta^{i-1}(f)w_i \in \mathbb{k}[V, W]^G$ , we define the multidegree of  $h$  to be that of  $f$ . Since  $G$  preserves the multidegree, this is the same as the multidegree of  $\Delta^{i-1}(f)$  for all  $i \leq s(h)$ . Then the analogue of this result for covariants is the following:

**Proposition 4.** *Let  $h$  be a covariant of multidegree  $d_1, d_2, \dots, d_m$  with  $d_j > p - n_j$  for some  $j$ . Then there exists a covariant  $h_1$  and a transfer covariant  $h_2$  such that  $h = N_j h_1 + h_2$ .*

*Proof.* We proceed by induction on the support  $s(h)$  of  $h$ . If  $s(h) = 1$ , then by Proposition 2, we have that  $h = fw_1$  with  $f \in \mathbb{k}[V]^G$ . Then we can write  $f = qN_j + \Delta^{p-1}(t)$  for some  $q \in \mathbb{k}[V]^G$  and  $t \in \mathbb{k}[V]$ . Then both  $qw_1$  and  $\Delta^{p-1}(t)w_1$  are covariants by Proposition 2 and therefore  $h = qN_j w_1 + \Delta^{p-1}(t)w_1$  gives us the desired decomposition.

Now assume that  $s(h) = k$ . Then by Proposition 2 there exists  $f \in \mathbb{k}[V]$  such that

$$h = fw_1 + \Delta(f)w_2 + \cdots + \Delta^{k-1}(f)w_k,$$

with  $\Delta^k(f) = 0$ . Since  $\Delta^{k-1}(f) \in \mathbb{k}[V]^G$  and  $d_j > p - n_j$ , we can write  $\Delta^{k-1}(f) = qN_j + \Delta^{p-1}(t)$  for some  $q \in \mathbb{k}[V]^G$  and  $t \in \mathbb{k}[V]$ . It follows that  $qN_j$  is in the image of  $\Delta^{k-1}$ . But since multiplication by  $N_j$  preserves the isomorphism type of a module, it follows that  $q$  is in the image of  $\Delta^{k-1}$ . Write  $q = \Delta^{k-1}(f')$  with  $f' \in \mathbb{k}[V]$ . Set  $h_1 = f'w_1 + \Delta(f')w_2 + \cdots + \Delta^{k-1}(f')w_k$  and  $h_2 = \Delta^{p-k}(t)w_1 + \cdots + \Delta^{p-1}(t)w_k$ . Since  $\Delta^{k-1}(f') \in \mathbb{k}[V]^G$ ,  $h_1$  is a covariant by Proposition 2. Consider the covariant

$h' = h - N_j h_1 - h_2$ . Since  $\Delta^{k-1}(f) = \Delta^{p-1}(t) + \Delta^{k-1}(f')N_j$ , the support of  $h'$  is strictly smaller than the support of  $h$ . Moreover,  $h_2$  is a transfer covariant and so the assertion of the proposition follows by induction.  $\square$

We obtain the following upper bound for the Noether number of covariants:

**Proposition 5.**  $\beta(\mathbb{k}[V, W]^G) \leq \max(\beta(\mathbb{k}[V]^G, \mathbb{k}[V]), mp - \dim(V))$ .

*Proof.* Let  $h \in \mathbb{k}[V, W]^G$  with degree  $d > \max(\beta(\mathbb{k}[V]^G, \mathbb{k}[V]), mp - \dim(V) + 1)$ . Let  $(d_1, d_2, \dots, d_m)$  be the multidegree of  $h$ . Then we must have  $d_j > p - n_j$  for some  $j$ . Consequently we may apply Proposition 4, writing

$$h = N_j h_1 + h_2$$

where  $h_2$  is a transfer covariant. Since  $\deg(h_2) > \beta(\mathbb{k}[V]^G, \mathbb{k}[V])$ ,  $h_2$  is decomposable by Proposition 3, and so we have shown that  $h$  is decomposable.  $\square$

#### 4. LOWER BOUNDS

Indecomposable transfers are one method of obtaining lower bounds for  $\mathbb{k}[V]^G$ . The analogous result for covariants is:

**Lemma 6.** *Let  $n \geq 2$  and  $\Delta^{p-1}(f) \in \mathbb{k}[V]^G$  be a transfer which is indecomposable. Then the transfer covariant*

$$h = \Delta^{p-n}(f)w_1 + \dots + \Delta^{p-1}(f)w_n$$

*is indecomposable.*

*Proof.* Assume on the contrary that  $h$  is decomposable. Then there exist  $q_i \in \mathbb{k}[V]^G_+$  and  $h_i \in \mathbb{k}[V, W]^G$  for  $1 \leq i \leq t$  such that  $h = \sum_{1 \leq i \leq t} q_i h_i$ . Write  $h_i = h_{i,1}w_1 + \dots + h_{i,n}w_n$  for  $1 \leq i \leq t$ . Then we have  $\Delta^{p-1}(f) = \sum_{1 \leq i \leq t} q_i h_{i,n}$ . By Proposition 2 we have  $\Delta(h_{i,n-1}) = h_{i,n}$  and so  $h_{i,n} \in \mathbb{k}[V]^G_+$  because  $n \geq 2$ . It follows that  $\Delta^{p-1}(f) = \sum_{1 \leq i \leq t} q_i h_{i,n}$  is a decomposition of  $\Delta^{p-1}(f)$  in terms of invariants of strictly smaller degree, contradicting the indecomposability of  $\Delta^{p-1}(f)$ .  $\square$

**Corollary 7.** *Suppose  $n \geq 2$  and  $\beta(\mathbb{k}[V]^G) > \max(p, mp - \dim(V))$ . Then  $\beta(\mathbb{k}[V]^G) \leq \beta(\mathbb{k}[V, W]^G)$ .*

*Proof.* By [8, Lemma 2.12],  $\mathbb{k}[V]^G$  is generated by the norms  $N_1, N_2, \dots, N_m$ , invariants of degree at most  $mp - \dim(V)$ , and transfers. Since there exists an indecomposable invariant of degree  $\beta(\mathbb{k}[V]^G)$ , if the hypotheses of the Corollary hold, then  $\mathbb{k}[V]^G$  contains an indecomposable transfer with this degree. By Lemma 6,  $\mathbb{k}[V, W]^G$  contains a transfer covariant of degree  $\beta(\mathbb{k}[V]^G)$  which is indecomposable, from which the conclusion follows.  $\square$

#### 5. MAIN RESULTS

We are now ready to prove Theorem 1. Note that  $\mathbb{k}[V, V_1]^G$  is generated by over  $\mathbb{k}[V]^G$  by  $w_1$  alone, which has degree zero, and therefore  $\beta(\mathbb{k}[V, V_1]^G) = 0$ . For this reason we assume  $n \geq 2$  throughout.

*Proof.* Suppose first that  $n_j > 3$  for some  $j$ . Then by [6, Proposition 1.1(a)], we have

$$\beta(\mathbb{k}[V]^G) = m(p-1) + (p-2).$$

Since  $V$  is reduced we have  $\dim(V) \geq 2m$  and hence

$$\beta(\mathbb{k}[V]^G) > m(p-2) \geq mp - \dim(V).$$

Also,  $\beta(\mathbb{k}[V]^G) \geq 2p-3 > p$  since  $n_j \leq p$  for all  $j$ . Therefore the Corollary 7 implies that  $\beta(\mathbb{k}[V]^G) \leq \beta(\mathbb{k}[V, W]^G)$ . On the other hand, [6, Lemma 3.3] shows that the

top degree of  $\mathbb{k}[V]/\mathbb{k}[V]_+^G\mathbb{k}[V]$  is bounded above by  $m(p-1) + (p-2)$ . By the graded Nakayama Lemma it follows that  $\beta(\mathbb{k}[V], \mathbb{k}[V]^G) \leq m(p+1) + (p-2)$ . We have already shown that this number is at least  $mp - \dim(V) + 1$ , so by Proposition 5 we get that

$$\beta(\mathbb{k}[V, W]^G) \leq m(p-1) + (p-2) = \beta(\mathbb{k}[V]^G)$$

as required.

Now suppose that  $n_i \leq 3$  for all  $i$  and  $n_j = 3$  for some  $j$ . Then by [6, Proposition 1.1(b)], we have

$$\beta(\mathbb{k}[V]^G) = m(p-1) + 1.$$

Since  $V$  is reduced we have  $\dim(V) \geq 2m$  and hence

$$\beta(\mathbb{k}[V]^G) > m(p-2) \geq mp - \dim(V).$$

Also  $\beta(\mathbb{k}[V]^G) \geq 2p-1 > p$  provided  $m \geq 2$ . In that case Corollary 7 applies. If  $m = 1$  then Dickson [4] has shown that  $\mathbb{k}[V]^G = \mathbb{k}[x_1, x_2, x_3]^G$  is minimally generated by the invariants  $x_3, x_2^2 - 2x_1x_3 - x_2x_3, N, \Delta^{p-1}(x_1^{p-1}x_2)$ . It follows that  $\Delta^{p-1}(x_1^{p-1}x_2)$  is an indecomposable transfer, so by Lemma 6  $\mathbb{k}[V, W]^G$  contains an indecomposable transfer covariant of degree  $p = \beta(\mathbb{k}[V]^G)$ . In either case we obtain

$$\beta(\mathbb{k}[V, W]^G) \geq \beta(\mathbb{k}[V]^G).$$

On the other hand, by [9, Corollary 2.8],  $m(p-1) + 1$  is an upper bound for the top degree of  $\mathbb{k}[V]/\mathbb{k}[V]_+^G$ . By the same argument as before we get  $\beta(\mathbb{k}[V]^G, \mathbb{k}[V]) \leq m(p-1) + 1$ . We have already shown that this number is at least  $mp - \dim(V) + 1$ , so by Proposition 5 we get that

$$\beta(\mathbb{k}[V, W]^G) \leq m(p-1) + (p-2) = \beta(\mathbb{k}[V]^G)$$

as required.

It remains to deal with the case  $n_i = 2$  for all  $i$ , i.e.  $V = mV_2$ . We assume  $m \geq 2$ . In this case Campbell and Hughes [2] showed that  $\beta(\mathbb{k}[V]^G) = (p-1)m$ . As  $\dim(V) = 2m$  we have  $\beta(\mathbb{k}[V]^G) > m(p-2) = mp - \dim(V)$ . If  $m \geq 3$  or  $m = 2$  and  $p > 2$  then we have  $\beta(\mathbb{k}[V]^G) > p$  and Corollary 7 applies. In case  $m = 2 = p$ ,  $\mathbb{k}[V]^G = \mathbb{k}[x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}]^G$  is a hypersurface, minimally generated by  $\{x_{2,1}, N_1, x_{2,2}, N_2, \Delta^{p-1}(x_{1,1}x_{1,2})\}$ . In particular  $\Delta^{p-1}(x_{1,1}x_{1,2})$  is an indecomposable transfer, so by Lemma 6,  $\mathbb{k}[V, W]^G$  contains an indecomposable transfer covariant of degree 2. In both cases we get

$$\beta(\mathbb{k}[V, W]^G) \geq \beta(\mathbb{k}[V]^G).$$

On the other hand, by [9, Theorem 2.1], the top degree of  $\mathbb{k}[V]/\mathbb{k}[V]_+^G\mathbb{k}[V]$  is bounded above by  $m(p-1)$ . We have already shown this number is at least  $mp - \dim(V) + 1$ . Therefore by Proposition 5 we get  $\beta(\mathbb{k}[V, W]^G) \leq \beta(\mathbb{k}[V]^G)$  as required.  $\square$

*Remark 8.* The only reduced representation not covered by Theorem 1 is  $V = V_2$ . An explicit minimal set of generators of  $\mathbb{k}[V_2, W]^G$  as a module over  $\mathbb{k}[V_2]^G$  is given in [5], the result is

$$\beta(\mathbb{k}[V_2, W]) = n - 1.$$

This is the only situation in which the Noether number is seen to depend on  $W$ .

*Remark 9.* Suppose  $V$  is any  $\mathbb{k}G$ -module and  $W = W_1 \oplus W_2$  is a decomposable  $\mathbb{k}G$ -module. Then

$$\mathbb{k}[V, W]^G = (S(V^*) \otimes (W_1 \oplus W_2))^G = (S(V^*) \otimes W_1) \oplus (S(V^*) \otimes W_2)^G.$$

So  $\beta(\mathbb{k}[V, W]^G) = \max\{(\beta(\mathbb{k}[V, W_i]^G) : i = 1, 2)\}$ . Thus, the results of this paper can be used in principal to compute  $\beta(\mathbb{k}[V, W]^G)$  for arbitrary  $\mathbb{k}G$ -modules  $V$  and  $W$ .

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MIDDLESEX UNIVERSITY, THE BURROUGHS, HENDON, LONDON, NW4 4BT UK  
*E-mail address*: j.elmer@mdx.ac.uk

BILKENT UNIVERSITY, DEPARTMENT OF MATHEMATICS, CANKAYA, ANKARA, 06800 TURKEY  
*E-mail address*: sezer@fen.bilkent.edu.tr