ON THE DEPTH OF QUOTIENTS OF MODULAR INVARIANT RINGS BY TRANSFER IDEALS

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Abstract. Let G be a finite group, and V a finite dimensional vector space over a field $\Bbbk$ of characteristic dividing the order of G. Let $H \leq G$. The transfer map $\mathrm{Tr}^G_H : \Bbbk[V]^H \to \Bbbk[V]^G$ is an important feature of modular invariant theory. Its image is called a transfer ideal I^G_H of $\Bbbk[V]^G$, and this ideal, along with the quotients $\Bbbk[V]^G/I^G_H$ are widely studied.

In this article we study $\Bbbk[V]^G/I$, where I is any sum of transfer ideals. Our main result gives an explicit regular sequence of length $\dim(V^G)$ in $\Bbbk[V]^G/I$ when G is a p-group. We identify situations where this is sufficient to compute the depth of $\Bbbk[V]^G/I$, in particular recovering a result of Totaro. We also study the cases where G is cyclic or isomorphic to the Klein 4 group in greater detail. In particular we use our results to compute the depth of $\Bbbk[V]^G/I^G_{\{1\}}$ for an arbitrary indecomposable representation of the Klein 4 group.

1. Introduction

Let $\Bbbk$ be an infinite field and V a finite-dimensional $\Bbbk$-vector space, and $G \leq \mathrm{GL}(V)$ a finite group. Then the induced action on V^* extends to the symmetric algebra $\Bbbk[V] := S(V^*)$ by the formula

$$\sigma(f) = f \circ \sigma^{-1}$$

for $\sigma \in G$ and $f \in \Bbbk[V]$. The algebra of fixed points $\Bbbk[V]^G$ is called the *ring of invariants*, and is the central object of study in invariant theory.

Now let H be a subgroup of G. There is a well-defined $\Bbbk$-linear map

$$\mathrm{Tr}^G_H : \Bbbk[V]^H \to \Bbbk[V]^G$$
$$f \mapsto \sum_{\sigma \in G/H} \sigma f$$

which is called the *relative transfer* from H to G. If H is trivial we call this the *transfer* to G and denote it by Tr^G.

If $[G : H]$ is nonzero in $\Bbbk$, then it is easy to show that Tr^G_H is surjective. In particular, Tr^G is a surjective map if $|G|$ is not divisible by the characteristic of $\Bbbk$.

From now on suppose $\Bbbk$ is a field of positive characteristic p dividing $|G|$. Then if H is a p-subgroup of G, but not a Sylow-p-subgroup, the transfer Tr^G_H is not surjective and its image is a proper ideal of $\Bbbk[V]^G$. We denote this ideal by I^G_H. More generally, to any set $\mathcal{X}$ of subgroups of G we associate the ideal

$$I^G_{\mathcal{X}} = \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(\Bbbk[V]^X). \qquad (1)$$

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Transfer maps and their images are an important feature of modular invariant theory. For instance, a long-standing conjecture of Shank and Wehlau [7] states that if G is a p-group then $\Bbbk[V]^G$ is polynomial if and only if $I^G_{\{1\}}$ is a principal ideal. Other investigations concern the transfer ideal $I := I^G_{<P}$ where P is a Sylow-p-subgroup and $< P$ denotes the set of all proper subgroups of P. This can be shown to be independent of the choice of Sylow-p-subgroup. Notable are a conjecture of Wehlau et al [10] stating that $\Bbbk[V]^G/I$ is always generated by invariants of degree $\leq |G|$, and a recent result of Totaro [9] stating that $\Bbbk[V]^G/I$ is always a Cohen-Macaulay ring. Both of these conform to a general philosophy that $\Bbbk[V]^G/I$ behaves rather like a non-modular ring of invariants.

In the present article we consider quotient rings of the form $R^G_{\mathcal{X}} := \Bbbk[V]^G/I^G_{\mathcal{X}}$ where G is a p-group and $\mathcal{X}$ is any set of proper subgroups of G. Our main result (Proposition 1) gives an explicit regular sequence of length $\dim(V^G)$ in $R^G_{\mathcal{X}}$. In particular, we recover Totaro's result in the special case of a p-group by elementary means. We also give some new results on the depth of $R^G_{\mathcal{X}}$ where G is a cyclic p-group, and classify indecomposable representations V of the Klein 4-group G for which $\Bbbk[V]^G/I^G_{\{1\}}$ is Cohen-Macaulay.

2. A regular sequence

In this section let $\Bbbk$ be a field and $A = \oplus_{i\geq 0} A_i$ a commutative graded $\Bbbk$-algebra with $A_0 = \Bbbk$. Let M be a graded A-module. An *M-regular element* is an element $a \in A$ such that the map $\phi_a : M \to M$ defined by

$$\phi_a(m) = am$$

is injective. An *M-regular sequence* is a sequence $a_1, a_2, \ldots, a_r$ such that a_1 is M-regular and for all $i = 2, \ldots, r$, a_i is $M/(a_1, \ldots, a_{i-1})M$-regular.

An M-regular sequence is called *maximal* if it cannot be extended to a longer M-regular sequence. If A is Noetherian and M is finite, then it can be shown that all maximal M-regular sequences have the same length. Define $A_+ = \oplus_{i\geq 1} A_i$. Then we call the length of a maximal M-regular sequence in A_+ the *depth* of M. The depth of A is the depth of A as a module over itself.

For further results on depth and regular sequences we recommend [2].

Now let G be a p-group and adopt the notation introduced in the introduction. Let $v_1, \ldots, v_l$ be a basis of the fixed-point space V^G and extend to a basis $v_1, \ldots, v_n$ of V. Let $x_1, x_2, \ldots, x_n$ be the corresponding dual basis of V^*. Then $\Bbbk[V]$ is the polynomial ring generated by $x_1, x_2, \ldots, x_n$. For $i = 1, \ldots, l$ we define

$$N_i = N^G(x_i) = \prod_{\sigma \in G} \sigma x_i. \tag{2}$$

Notice that N_i is a monic polynomial of degree $|G|$ as a polynomial in x_i, and degree zero as a polynomial in x_j for any $j = 1, \ldots, i-1, i+1, \ldots, l$. To see the second statement, note that

$$N_i(v_j) = \prod_{\sigma \in G} (\sigma x_i) v_j = \prod_{\sigma \in G} x_i(\sigma^{-1} v_j) = \prod_{\sigma \in G} x_i(v_j) = 0.$$

Further, let $\mathcal{X}$ be a set of proper subgroups of G. Given $f \in \Bbbk[V]^G$, we denote by $\overline{f}$ the image of f in $R^G_{\mathcal{X}}$. With this notation we have the following which was proven for a cyclic group of prime order in [6, Theorem 12].

Proposition 1.

$$\overline{N_1}, \ldots, \overline{N_l}$$

is a regular sequence in $R^G_{\mathcal{X}}$.

Proof. We first show that $\overline{N_i}$ is regular on $R^G_{\mathcal{X}}$ for any $i = 1, \ldots, l$. Let $i \in \{1, \ldots, l\}$ and suppose there exists $f \in \Bbbk[V]^G$ such that $\overline{fN_i} = 0$ in $R^G_{\mathcal{X}}$. In turn this means for each $X \in \mathcal{X}$, there exists a polynomial g_X such that

$$fN_i = \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(g_X). \tag{3}$$

We view g_X as a polynomial in x_i with coefficients in $\Bbbk[x_1, \ldots, x_{i-1}, x_{i+1}, x_n]$. Since N_i is a monic polynomial in x_i, we may divide g_X by N_i, finding unique polynomials q_X and r_X such that

$$g_X = q_X N_i + r_X$$

where the x_i-degree of r_X is $< |G|$. As $v_i \in V^G$, the action of G on $\Bbbk[V]$ does not increase x_i-degrees so the uniqueness of this decomposition implies that $q_X, r_X \in \Bbbk[V]^X$ and so

$$fN_i = \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(q_X N_i + r_X) = \sum_{X \in \mathcal{X}} N_i \,\mathrm{Tr}^G_X(q_X) + \mathrm{Tr}^G_X(r_X).$$

This can be rearranged as

$$0 = (f - \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(q_X))N_i + \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(r_X).$$

Since $\deg_{x_i} \mathrm{Tr}^G_X(r_X) < |G|$ for each X, comparing the x_i-degrees in the above equation gives

$$f - \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(q_X)) = 0,$$

i.e. $f \in I^G_{\mathcal{X}}$, which shows that $\overline{f} = 0$ as desired. Therefore, $\overline{N_i}$ is regular on $R^G_{\mathcal{X}}$.

Now assume $1 < j \leq l$ and $\overline{N_1}, \ldots, \overline{N_{j-1}}$ is a regular sequence in $R^G_{\mathcal{X}}$. Suppose $\overline{N_j}$ is not regular on $R^G_{\mathcal{X}}/(\overline{N_1}, \ldots, \overline{N_{j-1}})R^G_{\mathcal{X}}$. This means there exist $f_1, \ldots, f_j \in \Bbbk[V]^G$ such that

$$\overline{f_1 N_1} + \ldots + \overline{f_j N_j} = 0$$

in $R^G_{\mathcal{X}}$. This in turn means there exists, for each $X \in \mathcal{X}$, an element $g_X \in \Bbbk[V]^X$ such that

$$f_1 N_1 + \ldots + f_j N_j = \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(g_X). \tag{4}$$

We now divide each f_i (for $i = 2 \ldots j$) by N_1 with remainder, writing

$$f_i = q_{i,1} N_1 + r_{i,1}.$$

We obtain for the right-hand side

$$f_1 N_1 + ((q_{2,1})N_1 + r_{2,1})N_2 + ((q_{3,1})N_1 + r_{3,1})N_3 + \ldots + ((q_{j,1})N_1 + r_{j,1})N_j.$$

By collecting together all terms divisible by N_1 to replace f_1, and replacing each f_i by $r_{i,1}$, we can assume f_i has degree $< |G|$ as a polynomial in x_1. Continuing this process, dividing by each N_i $(i = 2, \ldots, l)$ in turn and relabeling if necessary, allows us to assume f_i has degree $< |G|$ in all variables $x_1, \ldots, x_{i-1}$. Further, for each $X \in \mathcal{X}$ we write

$$g_X = q_{X,1} N_1 + q_{X,2} N_2 + \ldots q_{X,j} N_j + r_X$$

where the degree of $q_{X,i}$ is $< |G|$ in each variable $x_1, \ldots, x_{i-1}$ and the degree of r_X is $< |G|$ in each of $x_1, \ldots, x_j$. With further relabeling (bringing terms $\mathrm{Tr}^G_X(q_{X,i})N_i$ to the left hand side) we obtain the expression

$$f_1 N_1 + \ldots + f_j N_j = \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(r_X) \tag{5}$$

in which each f_i has degree $< |G|$ as a polynomial in each variable $x_1, \ldots, x_{i-1}$ and each r_X has degree $< |G|$ as a polynomial in each variable $x_1, \ldots, x_j$.

Now notice that that, as a polynomial in x_1, the term $f_1 N_1$ has degree $\geq |G|$ unless $f_1 = 0$. All other terms in the expression have x_1-degree $< |G|$. This shows that $f_1 = 0$. Considering degrees in $x_2, \ldots, x_{j-1}$ now shows that $f_2 = f_3 = \ldots = f_{j-1} = 0$, too. So we're left with

$$f_j N_j = \sum_{X \in \mathcal{X}} \mathrm{Tr}^G_X(r_X).$$

Since we already know that N_j is regular on $R^G_{\mathcal{X}}$, we must have $f_j = 0$. This completes the proof.

□

3. Upper Bounds

Proposition 1 implies that

$$\mathrm{depth}(R^G_{\mathcal{X}}) \geq \dim(V^G)$$

for any set $\mathcal{X}$ of proper subgroups of the p-group G. In order to compute the exact depth of $R^G_{\mathcal{X}}$, we need an upper bound for depth. It is well-known that, for any Noetherian $\Bbbk$-algebra A, $\mathrm{depth}(A) \leq \dim(A)$, see e.g. [2, Proposition 1.2.12]. With this in mind, we want to compute $\dim(R^G_{\mathcal{X}})$.

As a first step, we note that $I^G_{\mathcal{X}} = I^G_{\overline{\mathcal{X}}}$, where $\overline{\mathcal{X}}$ is obtained from $\mathcal{X}$ by including all subgroups of every $X \in \mathcal{X}$ and their G-conjugates. This follows easily from e.g. [4, Equation (1)]. Now we have

Proposition 2.

$$\dim(R^G_{\mathcal{X}}) = \max\{\dim(V^Q) : Q \notin \overline{\mathcal{X}}\}.$$

Proof. Note that $\Bbbk[V]^G$ can be interpreted as the ring of regular functions on the quotient V/G. Then by definition we have

$$\dim(R^G_{\mathcal{X}}) = \dim(\mathcal{V}(I^G_{\mathcal{X}}))$$

where, for an ideal J of $\Bbbk[V]^G$, $\mathcal{V}(J)$ denotes the set of G-orbits vanishing on J. By [4, Proposition 12.4(iii)] we have

$$\mathcal{V}(I^G_{\mathcal{X}}) = i^*_G\Big(\bigcup_{Q \leq G : Q \notin \overline{\mathcal{X}}} V^Q\Big)$$

where i_G is the inclusion map $\Bbbk[V]^G \hookrightarrow \Bbbk[V]$ and $i^*_G : V \to V/G$ is its dual. As the dimension of an algebraic variety is the largest dimension of an irreducible component, the result follows. □

Now in case $\mathcal{X} = \{X : X < G\}$ we get

$$\dim(V^G) \leq \mathrm{depth}(R^G_{\mathcal{X}}) \leq \dim(R^G_{\mathcal{X}}) = \dim(V^G).$$

Consequently we obtain

Corollary 3 (Totaro)**.** *$R^G_{<G}$ is a Cohen-Macaulay ring.*

For stronger upper bounds, we use the following, which is inspired by [5, Theorem 3.17].

Proposition 4. *Let $\mathcal{X}$ be a family of subgroups of G. Let $K \leq H$ for all $H \in \mathcal{X}$. If $((\bigcap_{H\in\mathcal{X}} I_K^H) \setminus I_K^G) \cap \Bbbk[V]^G \neq \emptyset$, then we have*

$$\operatorname{depth}(R_K^G) \leq \dim(R_{\mathcal{X}}^G).$$

Proof. Let $H \in \mathcal{X}$ and let $f \in (I_K^H \setminus I_K^G) \cap \Bbbk[V]^G$. Then $f = \operatorname{Tr}_K^H(g)$ for some $g \in \Bbbk[V]^K$. For $h \in \Bbbk[V]^H$ we have

$$\operatorname{Tr}_K^G(gh) = \operatorname{Tr}_H^G(\operatorname{Tr}_K^H(gh)) = \operatorname{Tr}_H^G(h \operatorname{Tr}_K^H(g)) = \operatorname{Tr}_H^G(hf) = f \operatorname{Tr}_H^G(h).$$

This shows that I_H^G annihilates $f + I_K^G \in \Bbbk[V]^G/I_K^G$. It follows that $I_{\mathcal{X}}^G$ annihilates $f + I_K^G \in \Bbbk[V]^G/I_K^G$. Therefore $I_{\mathcal{X}}^G$ is contained in one of the associated primes of $\Bbbk[V]^G/I_K^G$ in $\Bbbk[V]^G$. Since the depth of a module is smaller than the coheights of its associated primes, see [2, Proposition 1.2.13], the result follows. □

4. Cyclic groups

In this section G denotes a cyclic group of order p^r. There are exactly p^r indecomposable $\Bbbk G$-modules $V_1, \ldots, V_{p^r}$ and each indecomposable module V_n is afforded by a Jordan block of size n. We fix a generator σ of G. We choose a basis $e_1, e_2, \ldots, e_n$ for V_n such that the action of σ is given by $\sigma(e_i) = e_i + e_{i-1}$ for $2 \leq i \leq n$ and $\sigma(e_1) = e_1$. Define $\Delta = \sigma - 1$. We consider the subgroups $K \subseteq H$ of G generated by σ^{p^a} and σ^{p^b}, respectively. We identify a case when the premise of Proposition 4 is attained.

Lemma 5. *Assume that $\Bbbk[V]$ contains a summand which is isomorphic to V_n with $p^a - p^b + 1 \leq n \leq p^a - 1$. Then $(I_K^H \setminus I_K^G) \cap \Bbbk[V]^G \neq \emptyset$.*

Proof. Clearly, $e_1 \in V_n^G$. We demonstrate that $e_1 \in (I_K^H \setminus I_K^G)$. Note that Tr_K^H applies to K-invariants and a basis element e_i is in V_n^K if and only if $\sigma^{p^a}(e_i) = e_i$, or equivalently $\Delta^{p^a}(e_i) = 0$. It follows that $V_n^K = V_n$ since $n \leq p^a - 1$. The set $1, \sigma^{p^b}, \sigma^{2p^b}, \ldots, \sigma^{p^a - p^b}$ is a complete set of representatives for H/K. Therefore we have

$$\begin{aligned}\operatorname{Tr}_K^H =&(1 + \sigma^{p^b} + \sigma^{2p^b} + \cdots + \sigma^{p^a - p^b})\\ =&(1 + \sigma + \sigma^2 + \cdots + \sigma^{p^{a-b}-1})^{p^b}\\ =&(\Delta^{p^{a-b}-1})^{p^b} = \Delta^{p^a - p^b}.\end{aligned}$$

Then by assumption on n, we get $e_1 \in I_K^H$ because $\operatorname{Tr}_K^H(e_{p^a - p^b + 1}) = e_1$ and $e_{p^a-p^b+1} \in V_n^K$. Similarly, $1, \sigma, \ldots, \sigma^{p^a - 1}$ is a complete set of representatives for G/K and so $\operatorname{Tr}_K^G = (1 + \sigma + \sigma^2 + \cdots + \sigma^{p^a-1}) = \Delta^{p^a-1}$. It follows that $e_1 \notin I_K^G$ because $\Delta^{p^a-1}(V_n) = 0$ by assumption on n. □

For subgroups $K \subseteq H$ in G we have the decomposition $\operatorname{Tr}_K^G = \operatorname{Tr}_H^G \circ \operatorname{Tr}_K^H$, so $I_K^G \subseteq I_H^G$. It follows that for a collection $\mathcal{X}$ of subgroups, we have $I_{\mathcal{X}}^G = I_H^G$, where H is the largest subgroup in $\mathcal{X}$. So we consider just subgroups of G, rather than collections of subgroups.

Proposition 6. *Let K be a subgroup of G, generated by σ^{p^a}. If there is $0 < b \leq a$ such that there is an indecomposable summand V_n of $\Bbbk[V]$ with $p^a - p^b + 1 \leq n \leq p^a - 1$, then*

$$\operatorname{depth}(R_K^G) \leq \dim(V^{\sigma^{p^{b-1}}}) \leq p^{b-1} \dim(V^G).$$

Proof. Let H denote the subgroup of G generated by σ^{p^b}. Then from the previous lemma we have that $(I_K^H \setminus I_K^G) \cap \Bbbk[V]^G \neq \emptyset$. So Proposition 4 applies for $\mathcal{X} = \{H\}$ and we get that $\operatorname{depth}(R_K^G) \leq \dim(R^G/I_H^G)$. On the other hand $\dim(R^G/I_H^G) = \max\{\dim(V^Q) : H \subsetneqq Q\}$ by Proposition 2. Since G is cyclic, we have that $\max\{\dim(V^Q) : H \subsetneq Q\} = \dim(V^{\sigma^{p^{b-1}}})$ as desired. For the second inequality, note that for each summand V_m of V with basis $e_1, e_2, \ldots, e_m$, V_m^G is spanned by e_1 and $V_m^{\sigma^{p^{b-1}}}$ is spanned by $e_1, \ldots, e_k$, where $k = \min(p^{b-1}, m)$. □

We note an application for cyclic 2-groups.

Corollary 7. *Let G be a cyclic 2-group of order 2^r and let $V = V_t$ be the indecomposable module of dimension $t \leq 2^r$, where t is odd. Then* $\operatorname{depth}(R^G_{\{1\}}) = \dim(V^G)$.

Proof. Note that the d-th symmetric power $S^d(V_t)$ is isomorphic to $\Omega^{-d}(\wedge^d(V_{2^r-t}))$ modulo induced modules, by [8, Corollary 3.12]. Putting $d = 2^r - t$ we get $\wedge^d(V_{2^r-t}) = V_1$ and as d is odd we have $\Omega^{-d}(V_1) = V_{2^r-1}$. Therefore, V_{2^r-1} is a summand of the d-th homogeneous component of $\Bbbk[V_t]$. So applying the previous proposition for $b = 1$ and trivial K yields $\operatorname{depth}(R^G_{\{1\}}) \leq \dim(V^G)$. The reverse inequality follows from Proposition 1. □

5. The Klein 4-group

In this section we consider the action of the group $G = \langle \sigma_1, \sigma_2 \rangle \cong C_2 \times C_2$ on an indecomposable $\Bbbk G$-module V, where $\Bbbk$ is a field of characteristic 2. For each V, the depth of $\Bbbk[V]^G$ was computed in [3]. Further, if $\mathcal{X}$ contains all three proper subgroups of G, $R^G_{\mathcal{X}}$ is Cohen-Macaulay by Corollary 3. We will use the results of previous sections to compute $\operatorname{depth}(R^G_{\{1\}})$, for each indecomposable representation V of G. Note that the results of Section 3 imply

$$\dim(V^G) \leq \operatorname{depth}(R^G_{\{1\}}) \leq \dim(R^G_{\{1\}}) = \max_{1<Q<G} \dim(V^Q) \tag{6}$$

and $R^G_{\{1\}}$ is Cohen-Macaulay if and only if the second inequality is an equality.

The classification of indecomposable modules for this group is given in [1], and it is from this source we obtain our notation. To summarize, if V is an indecomposable $\Bbbk G$-module then V is isomorphic to a module in the following list:

- An even-dimensional representation of dimension n, denoted $V_{n,\lambda}$, where $\lambda \in \Bbbk \cup \{\infty\}$;
- The representations $\Omega^m(\Bbbk)$ where $m \in \mathbb{Z}$ and Ω represents the Heller shift operator. This representation has dimension $n = 2|m| + 1$.
- The regular representation $\Bbbk G$;

We analyse each case separately.

5.1. **Even dimensions, not projective.** First suppose $V \cong V_{n,\lambda}$ where $\lambda \in \Bbbk$. Write $n = 2m$. Then we can choose a basis $\{v_1, v_2, \ldots, v_m, w_1, \ldots, w_m\}$ with respect to which the action of G is as follows:

$$\sigma_i(v_j) = v_j \qquad i = 1, 2$$
$$\sigma_1(w_i) = w_i + v_i$$
$$\sigma_2(w_i) = w_i + \lambda v_i + v_{i-1}$$

where we set $v_0 = 0$. $V_{n,\infty}$ is obtained from $V_{n,0}$ by swapping the roles of σ_1 and σ_2. Note that, since the three modules $V_{n,0}$, $V_{n,1}$ and $V_{n,\infty}$ are linked by a group automorphism, they have isomorphic invariant rings, and in particular $\operatorname{depth}(R^G_{\{1\}})$ is the same for all three modules. So from now on we assume $\lambda \in \Bbbk$ and λ is invertible.

Now it is clear that $\dim(V^G) = m$. Moreover, if $\lambda \neq 1$ then $\dim(V^Q) = m$ for each maximal subgroup of G. So we obtain immediately from Equation 6 that $\operatorname{depth}(R^G_{\{1\}}) = m$ and this quotient ring is Cohen-Macaulay.

In case $\lambda = 1$ we have $\dim(V^{\sigma_1\sigma_2}) = m+1$. So $\dim(R^G_{\{1\}}) = m+1$. We claim that $\mathrm{depth}(R^G_{\{1\}}) = m$, so this quotient is not Cohen-Macaulay. Note that $V_{2,1}$ is not faithful, so we may assume $n \geq 4$.

To see this, consider the basis $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ of V^*, dual to the given basis of V. The action of G on this basis is given by

$$\sigma_i(y_j) = y_j \qquad i = 1,2$$
$$\sigma_1(x_i) = x_i + y_i$$
$$\sigma_2(x_i) = x_i + y_i + y_{i+1}$$

where $y_{m+1} = 0$.

Now we have
$$y_m = \mathrm{Tr}^{\langle\sigma_1\sigma_2\rangle}(x_{m-1}) \in I^{\langle\sigma_1\sigma_2\rangle}_{\{1\}}.$$
Moreover $y_m \in \Bbbk[V]^G$ and $y_m \notin I^G_{\{1\}}$ because V and hence V^* is not projective. Applying Proposition 4 with $\mathcal{X} = \{\langle\sigma_1\sigma_2\rangle\}$ and K trivial brings
$$\mathrm{depth}(R^G_{\{1\}}) \leq \dim(R^G_{\langle\sigma_1\sigma_2\rangle}) = m.$$
Combining this with Equation 6 proves the claim.

5.2. **Odd dimensions - negative Heller shift.** Suppose V is isomorphic to $\Omega^m(\Bbbk)$ where $m < 0$ (for $m = 0$ this representation is not faithful). Then we can choose a basis $\{v_1, v_2, \ldots, v_{m+1}, w_1, \ldots, w_m\}$ with respect to which the action of G is as follows:

$$\sigma_i(v_j) = v_j \qquad i = 1,2$$
$$\sigma_1(w_i) = w_i + v_i$$
$$\sigma_2(w_i) = w_i + v_{i+1}$$

Now its clear that $\dim(V^G) = m+1$. Moreover, $\dim(V^Q) = m+1$ for each maximal subgroup of G. So we obtain immediately from Equation 6 that $\mathrm{depth}(R^G_{\{1\}}) = m+1$ and this quotient ring is Cohen-Macaulay.

5.3. **Odd dimensions - positive Heller shift.** Suppose V is isomorphic to $\Omega^m(\Bbbk)$ where $m > 0$. Then we can choose a basis $\{v_1, v_2, \ldots, v_m, w_1, \ldots, w_{m+1}\}$ with respect to which the action of G is as follows:

$$\sigma_i(v_j) = v_j \qquad i = 1,2$$
$$\sigma_1(w_i) = w_i + v_i$$
$$\sigma_2(w_i) = w_i + v_{i-1}$$

where we set $v_{m+1} = v_0 = 0$.

Now it's clear that $\dim(V^G) = m$. Further, $\dim(V^{\sigma_1\sigma_2}) = m$ but $\dim(V^{\sigma_i}) = m+1$ for $i = 1,2$. So $\dim(R^G_{\{1\}}) = m+1$.

Assume $m \geq 2$. We claim that $\mathrm{depth}(R^G_{\{1\}}) = m$ and this quotient ring is not Cohen-Macaulay.

To see this, consider the basis $\{x_1, \ldots, x_m, y_1, \ldots, y_{m+1}\}$ of V^*, dual to the given basis of V. The action of G on this basis is given by

$$\sigma_i(y_j) = y_j \qquad i = 1,2$$
$$\sigma_1(x_i) = x_i + y_i$$
$$\sigma_2(x_i) = x_i + y_{i+1}.$$

Then we have
$$\mathrm{Tr}^{\langle\sigma_1\rangle}(x_m) = \mathrm{Tr}^{\langle\sigma_2\rangle}(x_{m-1}) = y_m.$$

Moreover $y_m \in \Bbbk[V]^G$ and $y_m \notin I^G_{\{1\}}$ because V and hence V^* is not projective. Applying Proposition 4 with $\mathcal{X} = \{\langle\sigma_1\rangle, \langle\sigma_2\rangle\}$ and K trivial brings

$$\operatorname{depth}(R^G_{\{1\}}) \leq \dim(R^G_{\mathcal{X}}) = \dim(V^{\sigma_1\sigma_2}) = m.$$

Combining this with Equation 6 proves the claim.

For $m = 1$ the ring of invariants is easy to compute: Let $N(x_1) = \prod_{g\in G} gx_1$. Note that $N(x_1)$ is x_1^4 modulo the ideal generated by y_1, y_2 in $\Bbbk[V]$. So the common zero set of the invariants $y_1, y_2, N(x_1)$ contains only the point zero, so this set is a homogeneous system of parameters for $\Bbbk[V]^G$. But since the product of their degrees is the group order, from a standard fact in invariant theory we get that $\Bbbk[V]^G = \Bbbk[y_1, y_2, N(x_1)]$. In particular, $\Bbbk[V]^G$ is a polynomial ring. Note that the ideal $\Bbbk[V]^G_+\Bbbk[V]$ is generated by y_1, y_2, x_1^4. Therefore $1, x_1, x_1^2, x_1^3$ forms a $\Bbbk$-basis for $\Bbbk[V]/\Bbbk[V]^G_+\Bbbk[V]$. Since Tr^G is a $\Bbbk[V]^G$-linear map, it follows that $I^G_{\{1\}}$ is generated by $\operatorname{Tr}^G(1), \operatorname{Tr}^G(x_1), \operatorname{Tr}^G(x_1^2), \operatorname{Tr}^G(x_1^3)$. But $\operatorname{Tr}^G(1) = \operatorname{Tr}^G(x_1) = \operatorname{Tr}^G(x_1^2) = 0$ and so $I^G_{\{1\}}$ is a principal ideal, generated by $\operatorname{Tr}^G(x_1^3) = y_1^2y_2 + y_1y_2$. Clearly this is a regular element of $\Bbbk[V]^G$. Therefore by [2, Theorem 2.1.3], $R^G_{\{1\}}$ is Cohen-Macaulay.

5.4. **The regular representation.** Let V_R denote the regular $\Bbbk G$-module with a basis v_1, v_2, v_3, v_4. The action of σ_1, σ_2 on V_R is given by the permutations $(1,2)(3,4)$ and $(1,3)(2,4)$, respectively. That is, we have $\sigma_i(v_j) = v_{\sigma_i(j)}$ for $i = 1,2$ and $1 \leq j \leq 4$. Note that $\dim(V^{\sigma_1}) = \dim(V^{\sigma_2}) = \dim(V^{\sigma_1\sigma_2}) = 2$ and so by Proposition 2 we have $\dim(R^G_{\{1\}}) = 2$. We demonstrate that $\operatorname{depth}(R^G_{\{1\}}) = 2$ as well. Consider the basis x_1, x_2, x_3, x_4 of V_R^* dual to the given basis of V_R. Since the inverse transpose of a matrix representing a product of disjoint tranpositions is equal to itself, the action of G on V_R^* is again given by

$$\sigma_i(x_j) = x_{\sigma_i(j)} \text{ for } i = 1,2 \text{ and } 1 \leq j \leq 4.$$

Since G permutes the variables, $\Bbbk[V_R]^G = \Bbbk[x_1, x_2, x_3, x_4]^G$ is generated as a vector space by orbit sums $o(m)$ of monomials in $\Bbbk[x_1, x_2, x_3, x_4]$. Note that $o(m) \in I^G_{\{1\}}$ for a monomial m if the orbit sum contains four monomials. Otherwise the monomial m has a non-trivial stabilizer and so $o(m) \notin I^G_{\{1\}}$. In this case the orbit sum $o(m)$ contains one or two monomials. We show that $N := x_1x_2x_3x_4$ and $H := o(x_1x_2) + o(x_1x_3) + o(x_1x_4) = x_1x_2 + x_3x_4 + x_1x_3 + x_2x_4 + x_1x_4 + x_2x_3$ form a regular sequence in $R^G_{\{1\}}$. Assume that $N(\sum_t c_t o(m_t)) \in I^G_{\{1\}}$, where $c_t \in \Bbbk$ and $o(m_t) \notin I^G_{\{1\}}$. So we may assume that each $o(m_t)$ contains one or two monomials. Note that the number of monomials in the orbits of m_t and Nm_t are the same and $N\sigma_i(m_t) = \sigma_i(Nm_t)$. Therefore $N(\sum_t c_t o(m_t)) = \sum_t c_t o(Nm_t)$ which is a sum of orbits with one or two monomials. It follows that $N(\sum_t c_t o(m_t)) \notin I^G_{\{1\}}$.

Next we show that H is a non-zero divisor in the quotient ring $R^G_{\{1\}}/NR^G_{\{1\}} = \Bbbk[x_1, x_2, x_3, x_4]^G/(I^G_{\{1\}}, N)$. Note that the ideal $(I^G_{\{1\}}, N)$ is spanned by orbit sums $o(m)$ of monomials where $o(m)$ has 4 monomials or m is divisible by N. Assume that $H(\sum_t c_t o(m_t)) \in (I^G_{\{1\}}, N)$ with $\sum_t c_t o(m_t) \notin (I^G_{\{1\}}, N)$. Since H is a homogeneous polynomial and $(I^G_{\{1\}}, N)$ is a homogeneous ideal, we may assume $\sum_t c_t o(m_t)$ is homogeneous as well. Each m_t has two monomials in its orbit (if it has one, then m_t is divisible by N), so each m_t is fixed by one of the $\sigma_1, \sigma_2, \sigma_1\sigma_2$. It follows that x_1, x_2, x_3, x_4 group in two pairs and the variables in the same pair appear with the same multiplicity in m_t. But m_t is not divisible by N so variables in one of the pairs do not appear at all in m_t. Therefore we may assume that m_t is equal to either $x_1^dx_2^d$, $x_1^dx_3^d$ or $x_1^dx_4^d$ for some positive integer d. So we have $H(c_1o(x_1^dx_2^d) + c_2o(x_1^dx_3^d) + c_3o(x_1^dx_4^d)) \in (I^G_{\{1\}}, N)$. Note that if $c_1 \neq 0$, then $x_1^{d+1}x_2^{d+1}$ appears in $Ho(x_1^d x_2^d)$. But $x_1^{d+1} x_2^{d+1}$ does not appear in $Ho(x_1^d x_3^d)$ or in $Ho(x_1^d x_4^d)$. So if $c_1 \neq 0$, then $x_1^{d+1} x_2^{d+1}$ appears in $H(c_1 o(x_1^d x_2^d) + c_2 o(x_1^d x_3^d) + c_3 o(x_1^d x_4^d))$. But $x_1^{d+1} x_2^{d+1}$ is fixed by σ_1 and it so it has two monomials in its orbit and we get that it does not appear in a polynomial in $(I_{\{1\}}^G, N)$. This gives $c_1 = 0$. Along the same lines one sees that $c_2 = 0$ and $c_3 = 0$.

5.5. **Decomposable Representations.** We let $V_1 \oplus V_2$ denote the direct sum of the $\Bbbk G$-modules V_1 and V_2, and let kV denote the direct sum of k copies of a $\Bbbk G$-module V. Assume that V is isomorphic to $c_{n,\lambda} V_{n,\lambda} + d_m \Omega^m(\Bbbk)$. Set $c = \sum c_{n,1}$ and $d = \sum_{m>0} d_m$. We prove that the Cohen-Macaulay defect of $R_{\{1\}}^G$ is bounded above by $\max\{c, d\}$.

Proposition 8. *Assume the notation of the previous paragraph. Then we have* $\dim(R_{\{1\}}^G) - \operatorname{depth}(R_{\{1\}}^G) \leq \max\{c, d\}$. *In particular, if V is a direct sum of modules of the form $V_{n,\lambda}$ with $\lambda \neq 1$ and $\Omega^m(\Bbbk)$ with $m < 0$, then $R_{\{1\}}^G$ is Cohen-Macaulay.*

Proof. Note that the dimensions of the fixed point spaces of σ_i and G are equal in all summands except $\Omega^m(\Bbbk)$ for $m > 0$ and $i = 1, 2$. Furthermore $\dim(\Omega^m(\Bbbk)^{\sigma_i}) = \dim(\Omega^m(\Bbbk)^G) + 1$ for $m > 0$ and $i = 1, 2$. It follows that $\dim(V^{\sigma_i}) = \dim(V^G) + d$ for $i = 1, 2$. On the other hand the dimensions of the fixed point spaces of $\sigma_1 \sigma_2$ and G are equal in all summands except $V_{n,1}$, and $\dim(V_{n,1}^{\sigma_1 \sigma_2}) = \dim(V_{n,1}^G) + 1$. Therefore $\dim(V^{\sigma_1 \sigma_2}) = \dim(V^G) + c$. So from Proposition 2 we get $\dim(R_{\{1\}}^G) = \dim(V^G) + \max\{c, d\}$. Since the depth of $R_{\{1\}}^G$ is at least $\dim(V^G)$ by Proposition 1, the result follows. □

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