# The messy maths of living things

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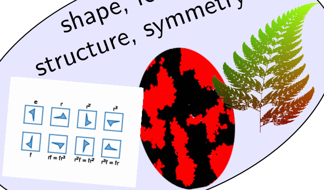
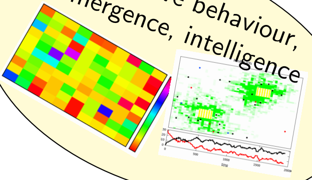
Summer Lecture, July 1st 2016

# Themes in this talk

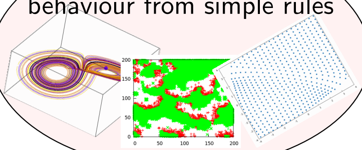
shape, form,
structure, symmetry



collective behaviour,
emergence, intelligence

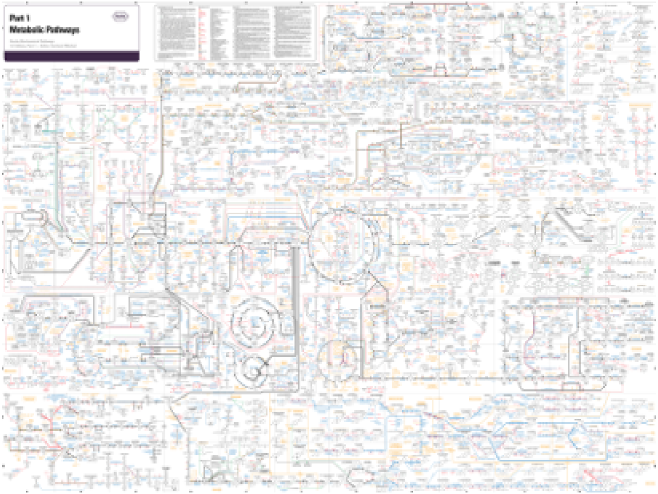


chaos, complex
behaviour from simple rules



# Life is complex

Thousands of linked chemical reactions take place in our cells.



# Story 1 How do cells die?

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Do they just rot away?

commons.wikimedia.org/w/index.php?curid=8685553

This can happen: it is called **necrosis**. It may happen after injury, for example.

All sorts of *mess* is released into the space around the cells.

The body may not cope well with the *clean-up*.

## Story 1 Self-sacrificing cells

**Apoptosis**: programmed cell death.

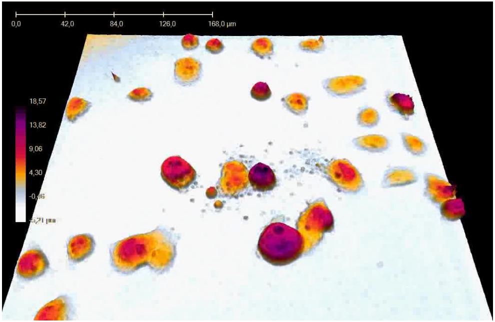
- Cells literally *switch themselves off*.
- This is much better than necrosis... the waste produced can be easily cleaned up by the body.
- But... when apoptosis goes wrong, all sorts of diseases can occur, including **cancer**.

**Q** What has apoptosis got to do with your fingers and toes?

**Q** How many cells die each day due to apoptosis in the average adult?

# Story 1 Treating cancer

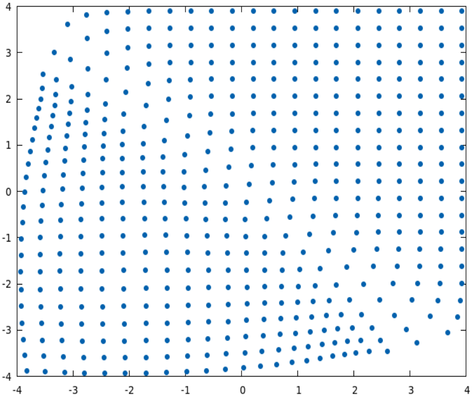
Here are some cancer cells being *induced* to undergo apoptosis. They are treated with a drug which *tells* them to switch off. The debris is then cleaned up by other cells.




http://www.cellimagelibrary.org/images/43705

## Story 1 Biological switches

Mathematicians are interested in which systems allow **switching** between different states.




Depending on where the system starts, it ends up in one state (a *fixed point*) or the other (a *periodic orbit*).

# Story 1 Can this chemical network switch?

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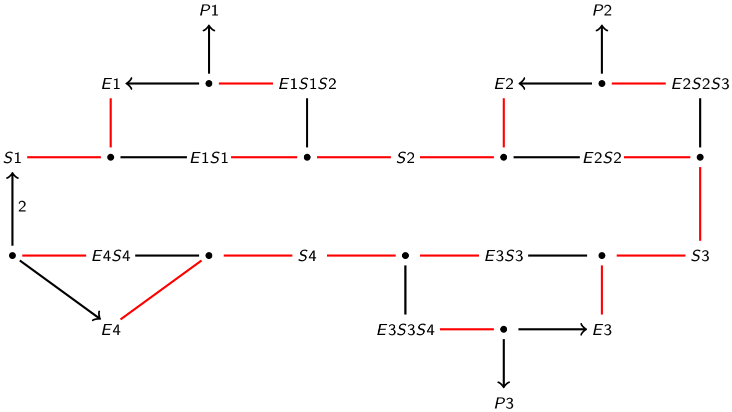


Using **analysis** and **graph theory**, we know this one can't! There is a well-developed theory on multistationarity in biological and chemical systems.

# Story 2 How animals walk

**Gait**: patterns of limb movement in an animal.

https://commons.wikimedia.org/w/index.php?curid=8567158

- In which order do the legs move?
- How is walking controlled?
- Do we think about each step?
- Who cares?

# Story 2 Central pattern generators

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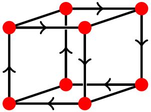
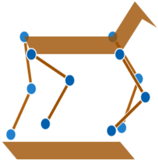
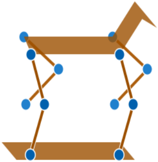
Small groups of connected cells called *central pattern generators* (CPGs) control gait. CPGs can generate rhythmic patterns without feedback from the body.

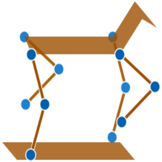




Walk

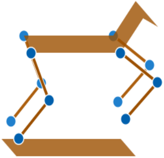


Pace



Trot

**Q** What is this gait? Can you think of other gaits?



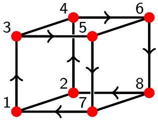
# Story 2 Gait: some equations

The previous animations were generated from some simple **differential equations**.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = c(x_i + y_i - \tfrac{1}{3}x^3) + \alpha(x_{i-2} - x_i) + \gamma(x_{i+\epsilon} - x_i), \qquad (i = 1, \ldots, 8)$$
$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = \tfrac{1}{c}(x_i - a + by_i) + \beta(y_{i-2} - y_i) + \delta(y_{i+\epsilon} - y_i)$$

(*M. Golubitsky, I. Stewart, P-L Buono and J.J. Collins*, Physica D 115 (1998))

- These equations have some **symmetry**. In fact they have the symmetry of this object:

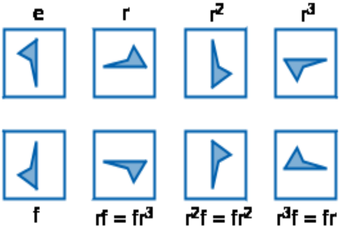



- The different gaits were obtained by varying the *parameters* $(a, b, c, \alpha, \beta, \gamma, \delta)$. So... the *same CPG* can generate many different patterns.

- Different gaits have different symmetries in space **and time**.

# Story 2 Symmetry

How do we *describe* the symmetries of an object?

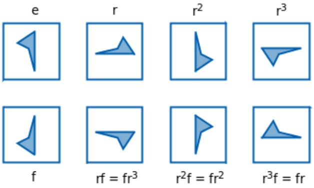



There are 8 transformations which preserve the square. They have various relationships between them, such as $rf = fr^3$.

**Q** How many transformations preserve a triangle?

# Story 2 Symmetry: group theory

Studying symmetry more carefully leads to an area of maths called **group theory**. Below is the group table for the square.




r = clockwise rotation by π/4

f = reflection in horizontal line

| | $e$ | $r$ | $r^2$ | $r^3$ | $f$ | $fr$ | $fr^2$ | $fr^3$ |
|---|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $r$ | $r^2$ | $r^3$ | $f$ | $fr$ | $fr^2$ | $fr^3$ |
| $r$ | $r$ | $r^2$ | $r^3$ | $e$ | $fr^3$ | $f$ | $fr$ | $fr^2$ |
| $r^2$ | $r^2$ | $r^3$ | $e$ | $r$ | $fr^2$ | $fr^3$ | $f$ | $fr$ |
| $r^3$ | $r^3$ | $e$ | $r$ | $r^2$ | $fr$ | $fr^2$ | $fr^3$ | $f$ |
| $f$ | $f$ | $fr$ | $fr^2$ | $fr^3$ | $e$ | $r$ | $r^2$ | $r^3$ |
| $fr$ | $fr$ | $fr^2$ | $fr^3$ | $f$ | $r^3$ | $e$ | $r$ | $r^2$ |
| $fr^2$ | $fr^2$ | $fr^3$ | $f$ | $fr$ | $r^2$ | $r^3$ | $e$ | $r$ |
| $fr^3$ | $fr^3$ | $f$ | $fr$ | $fr^2$ | $r$ | $r^2$ | $r^3$ | $e$ |

A group of transformations has various *subgroups*.

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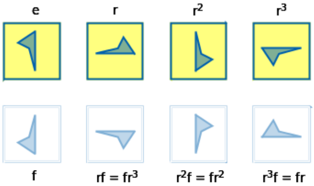


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| | e | r | $r^2$ | $r^3$ | f | fr | $fr^2$ | $fr^3$ |
|---|---|---|---|---|---|---|---|---|
| e | e | r | $r^2$ | $r^3$ | f | fr | $fr^2$ | $fr^3$ |
| r | r | $r^2$ | $r^3$ | e | $fr^3$ | f | fr | $fr^2$ |
| $r^2$ | $r^2$ | $r^3$ | e | r | $fr^2$ | $fr^3$ | f | fr |
| $r^3$ | $r^3$ | e | r | $r^2$ | fr | $fr^2$ | $fr^3$ | f |
| f | f | fr | $fr^2$ | $fr^3$ | e | r | $r^2$ | $r^3$ |
| fr | fr | $fr^2$ | $fr^3$ | f | $r^3$ | e | r | $r^2$ |
| $fr^2$ | $fr^2$ | $fr^3$ | f | fr | $r^2$ | $r^3$ | e | r |
| $fr^3$ | $fr^3$ | f | fr | $fr^2$ | r | $r^2$ | $r^3$ | e |

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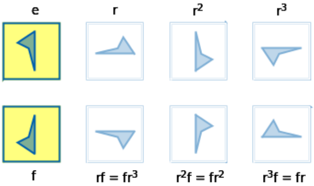


r = clockwise rotation by $\pi/4$

f = reflection in horizontal line

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|---|---|---|---|---|---|---|---|---|
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| r | r | $r^2$ | $r^3$ | e | $fr^3$ | f | fr | $fr^2$ |
| $r^2$ | $r^2$ | $r^3$ | e | r | $fr^2$ | $fr^3$ | f | fr |
| $r^3$ | $r^3$ | e | r | $r^2$ | fr | $fr^2$ | $fr^3$ | f |
| f | f | fr | $fr^2$ | $fr^3$ | e | r | $r^2$ | $r^3$ |
| fr | fr | $fr^2$ | $fr^3$ | f | $r^3$ | e | r | $r^2$ |
| $fr^2$ | $fr^2$ | $fr^3$ | f | fr | $r^2$ | $r^3$ | e | r |
| $fr^3$ | $fr^3$ | f | fr | $fr^2$ | r | $r^2$ | $r^3$ | e |

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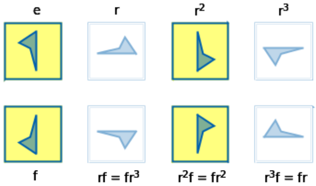


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|---|---|---|---|---|---|---|---|---|
| e | e | r | $r^2$ | $r^3$ | f | fr | $fr^2$ | $fr^3$ |
| r | r | $r^2$ | $r^3$ | e | $fr^3$ | f | fr | $fr^2$ |
| $r^2$ | $r^2$ | $r^3$ | e | r | $fr^2$ | $fr^3$ | f | fr |
| $r^3$ | $r^3$ | e | r | $r^2$ | fr | $fr^2$ | $fr^3$ | f |
| f | f | fr | $fr^2$ | $fr^3$ | e | r | $r^2$ | $r^3$ |
| fr | fr | $fr^2$ | $fr^3$ | f | $r^3$ | e | r | $r^2$ |
| $fr^2$ | $fr^2$ | $fr^3$ | f | fr | $r^2$ | $r^3$ | e | r |
| $fr^3$ | $fr^3$ | f | fr | $fr^2$ | r | $r^2$ | $r^3$ | e |

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# Story 2 Symmetry: group theory

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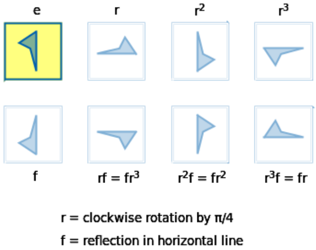
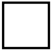


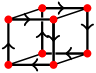

| | e | r | $r^2$ | $r^3$ | f | fr | $fr^2$ | $fr^3$ |
|---|---|---|---|---|---|---|---|---|
| e | e | r | $r^2$ | $r^3$ | f | fr | $fr^2$ | $fr^3$ |
| r | r | $r^2$ | $r^3$ | e | $fr^3$ | f | fr | $fr^2$ |
| $r^2$ | $r^2$ | $r^3$ | e | r | $fr^2$ | $fr^3$ | f | fr |
| $r^3$ | $r^3$ | e | r | $r^2$ | fr | $fr^2$ | $fr^3$ | f |
| f | f | fr | $fr^2$ | $fr^3$ | e | r | $r^2$ | $r^3$ |
| fr | fr | $fr^2$ | $fr^3$ | f | $r^3$ | e | r | $r^2$ |
| $fr^2$ | $fr^2$ | $fr^3$ | f | fr | $r^2$ | $r^3$ | e | r |
| $fr^3$ | $fr^3$ | f | fr | $fr^2$ | r | $r^2$ | $r^3$ | e |

A group of transformations has various *subgroups*.

# Story 2 Symmetry: group theory



The symmetry group of  is termed $D_4$. It has 8 elements.

The symmetry group of  is $\mathbb{Z}_4 \times \mathbb{Z}_2$. It has 8 elements.

$D_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ have the same size, but different structure in their subgroups.

By studying the gaits of quadrupeds, mathematicians postulated that their CPG has symmetry group $\mathbb{Z}_4 \times \mathbb{Z}_2$, and not $D_4$ (or anything simpler).

So mathematics led to a specific conclusion about the connections between cells in the CPG.

# Story 2 Gaits and group theory

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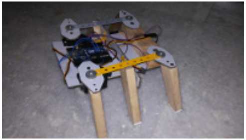
Different gaits have different symmetries in *space and time*. They correspond to different subgroups of a certain group (the product of the CPG symmetry group, and the circle group).

https://commons.wikimedia.org/w/index.php?curid=239615

A mixture of group theory and **dynamical systems** can give insight into the gaits of six-legged creatures and many-legged creatures too.

Many robots use (artificial) CPGs to generate different gaits efficiently.



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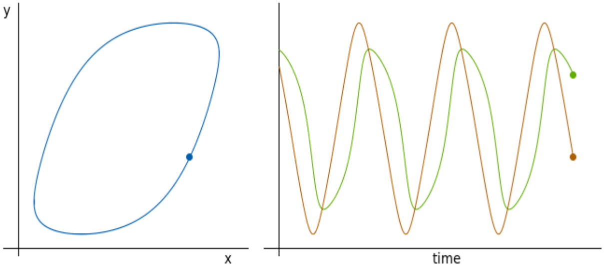
# Story 3 Oscillation and chaos

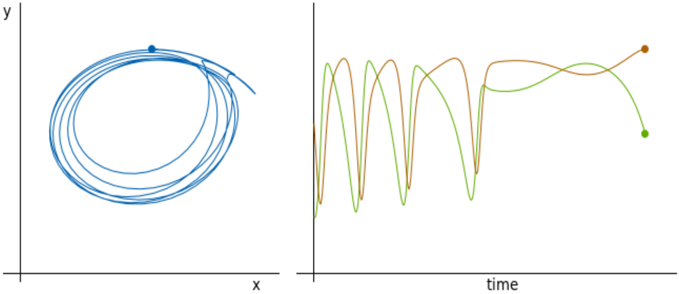
An oscillator is anything which changes periodically in time.
Simple differential equations can generate oscillation.




Biological **oscillators** include: neurons, heart cells, ecosystems...

**Q** In permanent light would you still feel sleepy every 24 hours?

# Story 3 Chaos




Something may appear like an oscillator...

...but do other strange stuff.

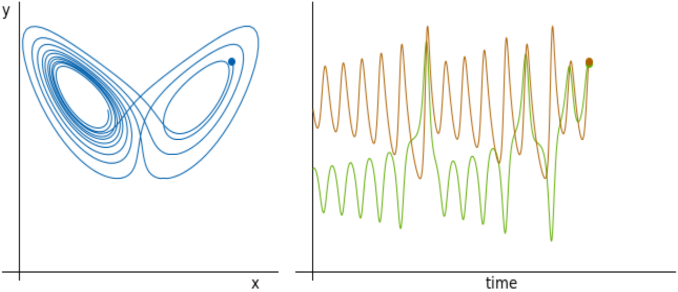
Simple differential equations can generate such behaviour.

This is an example of **chaos**.

# Story 3 Chaos: the Lorenz system



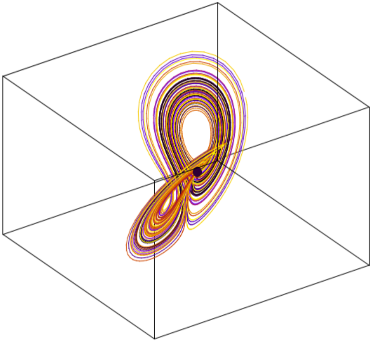
Here's a very famous chaotic system: the *Lorenz system*.




Three little differential equations generate this behaviour:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z\,.$$

We need 3D to see the beautiful geometry of what is happening:



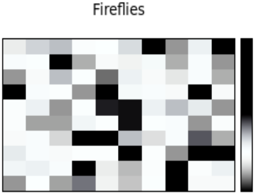
# Story 3 Chaos: the double pendulum

The normal pendulum is a boring and predictable oscillator...

...but the double pendulum is quite the opposite!

# Story 3 From synchronisation to chaos

Lots of interacting oscillators often like to **synchronise**. But synchronisation can break down in interesting ways...




**Q** Why do fireflies synchronise their flashing?

## Story 3 Chaos and biology



- Apparently chaotic behaviour has been observed both in computer simulations and experimental data from many biological systems.
- The transition between synchronisation and chaos may be important in understanding **heart attacks** and **epilepsy**.
- Chaos is probably much more common in biology than we realise, because it may be mistaken for noise.

# Story 4 Shape, pattern and form

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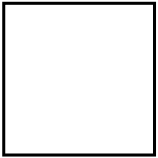
How do animals get their patterns?

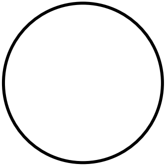
https://commons.wikimedia.org/w/index.php?curid=8567158

http://www.hyaenidae.org/Hyaenas-in-art-literature.html

https://commons.wikimedia.org/w/index.php?curid=1092469

Are some shapes more "mathematical" than others?





# Story 4 The mathematics of biological forms

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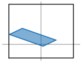
This fern leaf is generated on a computer with a few easy equations. It is an example of a **fractal**.

**Q** How long is the coast of Britain? (And what's the connection?!)

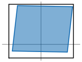
# Story 4 Generating the fern

$$\begin{pmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}$$



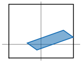
$$\uparrow$$

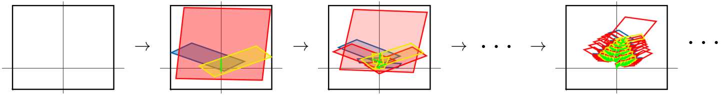
$$\begin{pmatrix} 0 & 0 \\ 0 & 0.16 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \longleftarrow \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}$$



$$\downarrow$$

$$\begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0.44 \end{pmatrix}$$





This is an example of an **iterated function system**.

# Story 4 Patterns from *reaction-diffusion systems*

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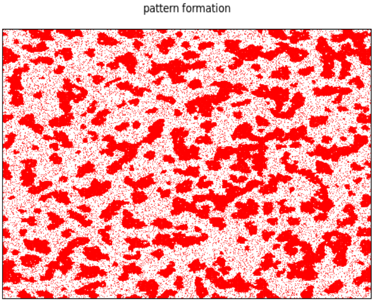



The details (spots or stripes, their size, shape, etc.) depend on the reactions, the diffusion rates and the **geometry**.

# Story 4 Biology and pattern formation

Middlesex University London

Animal coat patterns are quite well understood mathematically and are modelled with **partial differential equations**.

**Morphogenesis**: the biological process of developing shape

Understanding shape and pattern is not just mathematically interesting, but medically important too.

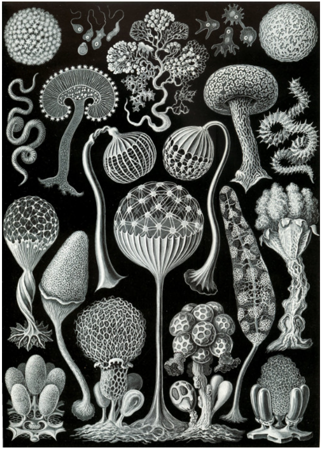
Embryologists need to understand how biological patterns develop. If something goes wrong, birth defects can result.

https://commons.wikimedia.org/w/index.php?curid=840032

**Q** Is the tip of an animal's tail generally stripy or spotty?

Are you one thing or many things?

# Story 5 Slime moulds: one or many?



Ernst Haeckel - Kunstformen der Natur (1904), plate 93: Mycetozoa. https://commons.wikimedia.org/w/index.php?curid=569696

*Slime moulds* are very strange: they live happily as single cells, but come together into complex organisms to reproduce.

Sometimes cooperation is so intense that we don't really know if something is one creature or many!

# Story 5 Intelligence is collective

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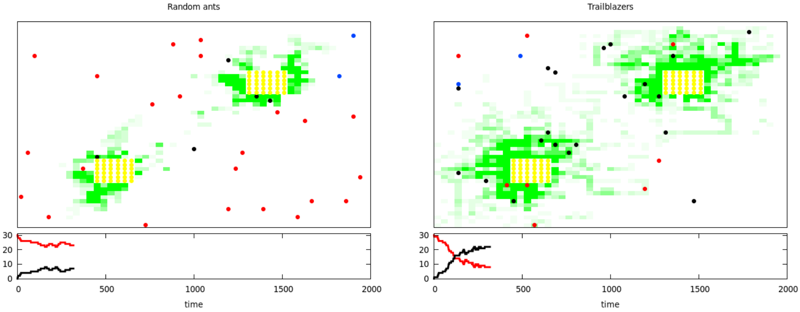


Individuals following very simple rules can together give rise to *intelligent* behaviour. This phenomenon is called **emergence**.

**Q** What amazing thing can slime moulds do?

# Story 5 Collective behaviour: predators and prey

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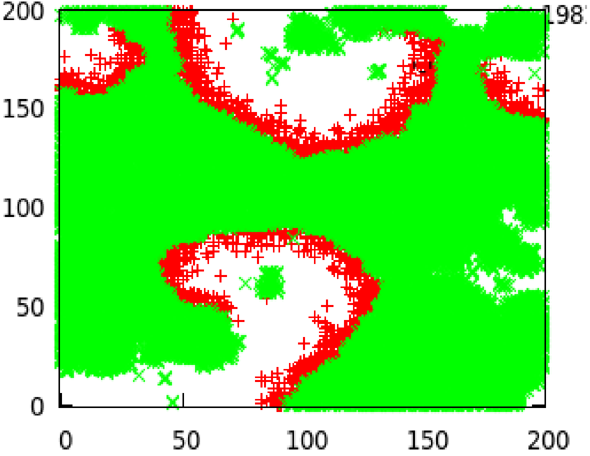
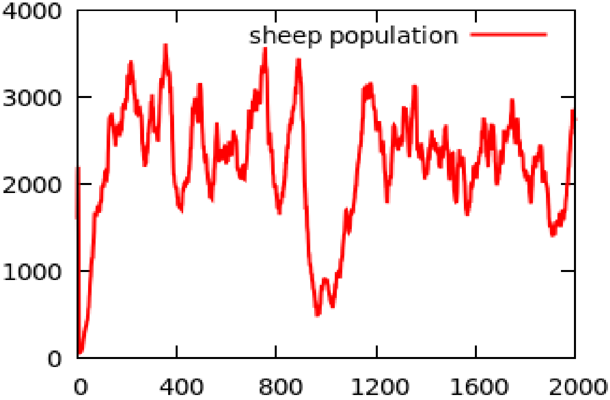



A predator-prey system. Waves and patterns are quite natural.

# Story 5 The sheep population




The total population shows noisy oscillations. Natural fluctuations can bring the population quite close to extinction.

# Story 5 Tenacious sheep or slow-growing grass

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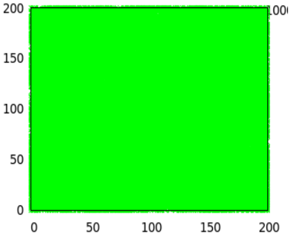


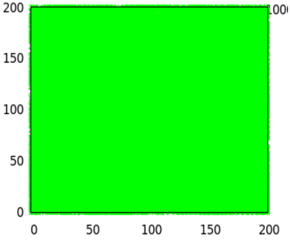


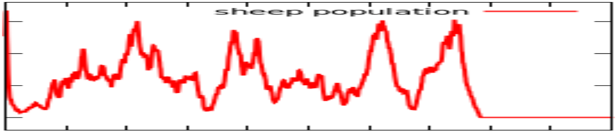


The fluctuations are more extreme, and extinction becomes likely.
A population of more successful individuals is more vulnerable!

# Some conclusions

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Maths is changing our understanding of **intelligence**, **pattern**, **cooperation**, **competition**, **evolution**, **complexity**, and of **life itself**.

It gives us new tools and a new, rich, language to describe the living world.

A lot of complex behaviours turn out to emerge from simple rules.

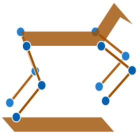
Future medicine will rely on these deeper understandings.

# Answers to the questions

**Q** What has apoptosis got to do with your fingers and toes?

**Q** In the average adult, how many cells die each day due to apoptosis (average adult)?

**Q** What is this gait? Can you think of other gaits?



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**Your fingers and toes are connected (webbed) in the womb. Apoptosis causes the webbing to disappear.**

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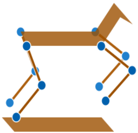

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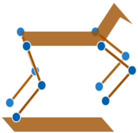

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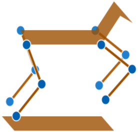


**That's a bound. Other gaits are a gallop (4 beats), a canter (3 beats), and unusual ones like the pronk (one beat).**

# Answers to the questions

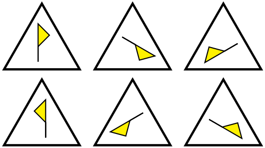
**Q** How many transformations preserve a triangle?

**Q** In permanent light would you still feel sleepy every 24 hours?

**Q** Why do fireflies synchronise their flashing?

# Answers to the questions

**Q** How many transformations preserve a triangle? **Six. The symmetry group of the triangle is called $D_3$ or $S_3$.**



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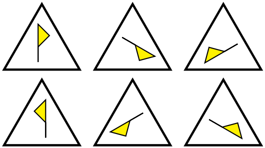

**Q** In permanent light would you still feel sleepy every 24 hours? **Yes, this is about our body clock, a natural biological oscillator. It has a cycle of about 24 hours.**

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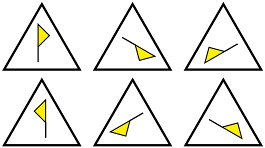


**Q** In permanent light would you still feel sleepy every 24 hours? **Yes, this is about our body clock, a natural biological oscillator. It has a cycle of about 24 hours.**

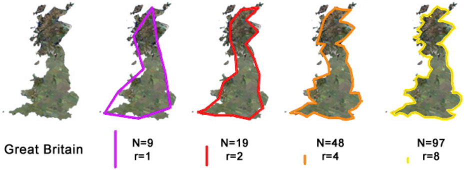
**Q** Why do fireflies synchronise their flashing? **The best hypothesis seems to be that they flash to attract mates: they are most visible if they flash in unison.**

# Answers to the questions

**Q** How long is the coast of Britain?

# Answers to the questions

Q How long is the coast of Britain?
**A trick question. Coastlines (like many biological surfaces) are approximately fractal in nature, so don't have a well-defined length. As you zoom in and see finer detail, the coastline appears longer and longer.**




From http://fractalfoundation.org/OFC/OFC-10-4.html

# Answers to the questions

**Q** Is the tip of an animal's tail generally stripy or spotty?

**Q** What amazing thing can slime moulds do?

# Answers to the questions

**Q** Is the tip of an animal's tail generally stripy or spotty?
**Stripy. Generally long thin geometries lead to stripes. This can be understood using differential equations.**

https://commons.wikimedia.org/w/index.php?curid=8621909

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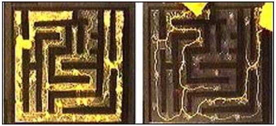
**Q** What amazing thing can slime moulds do?
**They can solve mazes!**



http://news.bbc.co.uk/1/hi/sci/tech/944790.stm

Thank you for listening!