MODULAR COVARIANTS OF CYCLIC GROUPS OF ORDER p

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Abstract. Let G be a cyclic group of order p and let V, W be $\Bbbk G$-modules. We study the modules of covariants $\Bbbk[V,W]^G = (S(V^*) \otimes W)^G$. For V indecomposable with dimension 2, and W an arbitrary indecomposable module, we show $\Bbbk[V,W]^G$ is a free $\Bbbk[V]^G$-module (recovering a result of Broer and Chuai [1]) and we give an explicit set of covariants generating $\Bbbk[V,W]^G$ freely over $\Bbbk[V]^G$. For V indecomposable with dimension 3, and W an arbitrary indecomposable module, we show that $\Bbbk[V,W]^G$ is a Cohen-Macaulay $\Bbbk[V]^G$-module (again recovering a result of Broer and Chuai) and we give an explicit set of covariants which generate $\Bbbk[V,W]^G$ freely over a homogeneous system of parameters for $\Bbbk[V]^G$. We also use our results to compute a minimal generating set for the transfer ideal of $\Bbbk[V]^G$ over a homogeneous sytem of parameters when V has dimension 3.

1. Introduction

Let G be a group, $\Bbbk$ a field, and V and W finite-dimensional $\Bbbk G$-modules on which G acts linearly. Then G acts on the set of functions $V \to W$ according to the formula

$$g \cdot \phi(v) = g\phi(g^{-1}v)$$

for all $g \in G$ and $v \in V$.

Classically, a *covariant* is a G-equivariant polynomial map $V \to W$. An *invariant* is the name given to a covariant $V \to \Bbbk$ where $\Bbbk$ denotes the trivial indecomposable $\Bbbk G$-module. If the field $\Bbbk$ is infinite, then the set of polynomial maps $V \to W$ can be identified with $S(V^*) \otimes W$, where the action on the tensor product is diagonal and the action on $S(V^*)$ is the natural extension of the action on V^* by algebra automorphisms. Then the natural pairing $S(V^*) \times S(V^*) \to S(V^*)$ is compatible with the action of G, and makes the invariants $S(V^*)^G$ a $\Bbbk$-algebra, and the covariants $(S(V^*) \otimes W)^G$ a $S(V^*)^G$-module.

If G is finite and the characteristic of $\Bbbk$ does not divide $|G|$, then Schur's lemma implies that every covariant restricts to an isomorphism of some direct summand of $S(V^*)$ onto W. Thus, covariants can be viewed as "copies" of W inside $S(V^*)$. Otherwise, the situation is more complicated.

The algebra of polynomial maps $V \to \Bbbk$ is usually written as $\Bbbk[V]$. In this article we will write $\Bbbk[V]^G$ for the algebra of G-invariants, and $\Bbbk[V,W]^G$ for the module of covariants. We are interested in the structure of $\Bbbk[V,W]^G$ as a $\Bbbk[V]^G$-module. Throughout, G denotes a finite group.

This question has been considered by a number of authors over the years. For example, Chevalley and Sheppard-Todd [2], [12] showed that if the characteristic of $\Bbbk$ does not divide $|G|$ and G acts as a reflection group on V, then $\Bbbk[V]^G$ is a polynomial algebra and $\Bbbk[V,W]^G$ is free. More generally, Eagon and Hochster

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[8] showed that if the characteristic of $\Bbbk$ does not divide $|G|$ then $\Bbbk[V,W]^G$ is a Cohen-Macaulay module (and $\Bbbk[V]^G$ a Cohen-Macaulay ring in particular). In the modular case, Hartmann [6] and Hartmann-Shepler [7] gave necessary and sufficient conditions for a set of covariants to generate $\Bbbk[V,W]^G$ as a free $\Bbbk[V]^G$-module, provided that $\Bbbk[V]^G$ is polynomial and $W \cong V^*$. Broer and Chuai [1] remove the restrictions on both W and $\Bbbk[V]^G$.

The present article is inspired by two particular results from [1], which we state here for convenience:

Proposition 1 ([1], Proposition 6)**.** *Let G be a finite group of order divisible by $p = char(\Bbbk)$ and let V, W be $\Bbbk G$-modules.*

(i) *Suppose* $\operatorname{codim}(V^G) = 1$. *Then $\Bbbk[V]^G$ is a polynomial algebra and $\Bbbk[V,W]^G$ is free as a graded module over $\Bbbk[V]^G$.*
(ii) *Suppose* $\operatorname{codim}(V^G) = 2$. *Then $\Bbbk[V,W]^G$ is a Cohen-Macaulay graded module over $\Bbbk[V]^G$.*

In the situation of (i) above, there is a method for checking a set of covariants generates $\Bbbk[V,W]^G$ over $\Bbbk[V]^G$, but no method of constructing generators. Meanwhile, in the situation of (ii), there exists a polynomial subalgebra A of $\Bbbk[V]^G$ over which $\Bbbk[V,W]^G$ is a free module. It is not clear how to find module generators, or to check that they generate $\Bbbk[V,W]^G$.

The purpose of this article is to work towards making these results constructive. We investigate certain modules of covariants for V satisfying (i) or (ii) above and G a cyclic group of order p.

2. Preliminaries

From this point onwards we let G be a cyclic group of order p and $\Bbbk$ a field of characteristic p. Let V and W be $\Bbbk G$-modules. We fix a generator σ of G. Recall that, up to isomorphism, there are exactly p indecomposable $\Bbbk G$-modules $V_1, V_2, \ldots, V_p$, where the dimension of V_i is i and each has fixed-point space of dimension 1. The isomorphism class of V_i is usually represented by a module of column vectors on which σ acts as left-multiplication by a single Jordan block of size i.

Suppose $W \cong V_n$. It is convenient to choose a basis $w_1, w_2, \ldots, w_n$ of W for which the action of G is given by

$$\begin{aligned}
\sigma w_1 &= w_1\\
\sigma w_2 &= w_2 - w_1\\
\sigma w_3 &= w_2 - w_2 + w_1\\
&\vdots\\
\sigma w_n &= w_n - w_{n-1} + w_{n-2} - \ldots \pm w_1.
\end{aligned}$$

(thus, the action of σ^{-1} is given by left-multiplication by a upper-triangular Jordan block). We do not (yet) choose a particular action on a basis for V, nor do we assume V is indecomposable; we let $v_1, v_2, \ldots, v_m$ be a basis of V and let $x_1, \ldots, x_m$ be the dual of this basis.

Note that $\Bbbk[V] = \Bbbk[x_1, x_2, \ldots, x_m]$, and a general element of $\Bbbk[V,W]$ is given by

$$\phi = f_1 w_1 + f_2 w_2 + \ldots + f_n w_n$$

where each $f_i \in \Bbbk[V]$. We define the **support** of ϕ by

$$\operatorname{Supp}(\phi) = \{i : f_i \neq 0\}.$$

The operator $\Delta = \sigma - 1 \in \Bbbk G$ will play a major role in our exposition. Δ is a *σ-twisted derivation* on $\Bbbk[V]$; that is, it satisfies the formula

$$\Delta(fg) = f\Delta(g) + \Delta(f)\sigma(g) \tag{1}$$

for all $f, g \in \Bbbk[V]$.

Further, using induction and the fact that σ and Δ commute, one can show Δ satisfies a Leibniz-type rule

$$\Delta^k(fg) = \sum_{i=0}^{k} \binom{k}{i} \Delta^i(f)\sigma^{k-i}(\Delta^{k-i}(g)). \tag{2}$$

A further result, which can be deduced from the above and proved by induction is the rule for differentiating powers:

$$\Delta(f^k) = \Delta(f)\left(\sum_{i=0}^{k-1} f^i \sigma(f)^{k-1-i}\right) \tag{3}$$

for any $k \geq 1$.

For any $f \in \Bbbk[V]$ we define the **weight** of f:

$$\mathrm{wt}(f) = \min\{i > 0 : \Delta^i(f) = 0\}.$$

Notice that $\Delta^{\mathrm{wt}(f)-1}(f) \in \ker(\Delta) = \Bbbk[V]^G$ for all $f \in \Bbbk[V]$. Another consequence of (2) is the following: let $f, g \in \Bbbk[V]$ and set $d = \mathrm{wt}(f), e = \mathrm{wt}(g)$. Suppose that

$$d + e - 1 \leq p.$$

Then

$$\Delta^{d+e-1}(fg) = \sum_{i=0}^{d+e-1} \binom{d+e-1}{i} \Delta^i(f)\sigma^{d+e-1-i}(\Delta^{d+e-1-i}(g)) = 0$$

since if $i < e$ then $d + e - 1 - i > d - 1$. On the other hand

$$\begin{aligned}\Delta^{d+e-2}(fg) &= \sum_{i=0}^{d+e-2} \binom{d+e-2}{i} \Delta^i(f)\sigma^{d+e-2-i}(\Delta^{d+e-2-i}(g)) \\ &= \binom{d+e-2}{i} \Delta^{d-1}(f)\sigma^{e-1}(\Delta^{e-1}(g)) \neq 0\end{aligned}$$

since $\binom{d+e-2}{i} \neq 0 \mod p$. We obtain the followng:

Proposition 2. *Let $f, g \in \Bbbk[V]$ with $\mathrm{wt}(f) + \mathrm{wt}(g) - 1 \leq p$. Then $\mathrm{wt}(fg) = \mathrm{wt}(f) + \mathrm{wt}(g) - 1$.*

Also note that

$$\Delta^p = \sigma^p - 1 = 0$$

which shows that $\mathrm{wt}(f) \leq p$ for all $f \in \Bbbk[V]^G$. Finally notice that

$$\Delta^{p-1} = \sum_{i=0}^{p-1} \sigma^i. \tag{4}$$

This is the *Transfer map*, a $\Bbbk[V]^G$-homomorphism $\mathrm{Tr}^G : \Bbbk[V] \to \Bbbk[V]^G$ which is well-known to invariant theorists.

Now we have a crucial observation concerning the action of σ on W: for all $i = 1, \ldots, n-1$ we have

$$\Delta(w_{i+1}) + \sigma(w_i) = 0 \tag{5}$$

and $\Delta(w_1) = 0$.

From this we obtain a simple characterisation of covariants:

Proposition 3. *Let*

$$\phi = f_1 w_1 + f_2 w_2 + \ldots + f_n w_n.$$

Then $\phi \in \Bbbk[V, W]^G$ *if and only if there exists* $f \in \Bbbk[V]$ *with weight* $\leq n$ *such that* $f_i = \Delta^{i-1}(f)$ *for all* $i = 1, \ldots, n$.

Proof. Assume $\phi \in \Bbbk[V, W]^G$. Then we have

$$\begin{aligned} 0 =& \Delta \left(\sum_{i=1}^n f_i w_i \right) \\ =& \sum_{i=1}^n \left(f_i \Delta(w_i) + \Delta(f_i)\sigma(w_i) \right) \\ =& \sum_{i=1}^{n-1} (\Delta(f_i) - f_{i+1})\sigma(w_i) + \Delta(f_n)\sigma(w_n) \end{aligned}$$

where we used (5) in the final step. Now note that

$$\sigma(w_i)) = w_i + (\text{terms in } w_{i-1}, w_{i-2}, \ldots, w_1)$$

for all $i = 1, \ldots, n$. Thus, equating coefficients of w_i, for $i = n, \ldots, 1$ gives

$$\Delta(f_n) = 0, \Delta(f_{n-1}) = f_n, \ldots, \Delta(f_2) = f_3, \Delta(f_1) = f_2.$$

Putting $f = f_1$ gives $f_i = \Delta^{i-1}(f)$ for all $i = 1, \ldots, n$ and $0 = \Delta^n(f)$ as required.

Conversely, suppose that

$$\phi = \sum_{i=1}^n \Delta^{i-1}(f) w_i$$

for some $f \in \Bbbk[V]$ with $\Delta^n(f) = 0$. Then we have

$$\begin{aligned} \Delta(\phi) &= \sum_{i=1}^n \Delta^{i-1}(f)\Delta(w_i) + \Delta^i(f)\sigma(w_i) \\ &= \sum_{i=2}^n (-\Delta^{i-1}(f)\sigma(w_{i-1}) + \Delta^i(f)\sigma(w_i)) + \Delta(f)\sigma(w_1) \qquad \text{by (5)} \\ &= \Delta^n(f)\sigma(w_n) \\ &= 0 \end{aligned}$$

as required.

□

Note that the support of any covariant is therefore of the form $\{1, 2, \ldots, i\}$ for some $i \leq n$. We will write

$$\text{Supp}(\phi) = i$$

if ϕ is a covariant and $\text{Supp}(\phi) = \{1, 2, \ldots, i\}$.

Introduce notation

$$K_n := \ker(\Delta^n)$$

and

$$I_n := \text{im}(\Delta^n).$$

These are $\Bbbk[V]^G$-modules lying inside $\Bbbk[V]$.

Now we can define a map

$$\Theta : K_n \to \Bbbk[V,W]^G$$

$$\Theta(f) = \sum_{i=1}^{n} \Delta^{i-1}(f) w_i. \tag{6}$$

Clearly Θ is an injective, degree-preserving map of $\Bbbk[V]^G$-modules, and Proposition 3 implies it is also surjective. We conclude that

Proposition 4. *K_n and $\Bbbk[V,W]^G$ are isomorphic as graded $\Bbbk[V]^G$-modules.*

From this point onwards we set $V = V_m$ and $W = V_n$, with the basis of V chosen so that

$$\begin{aligned}
\sigma x_1 &= x_1 + x_2,\\
\sigma x_2 &= x_2 + x_3,\\
\sigma x_3 &= x_3 + x_4.\\
&\vdots\\
\sigma x_m &= x_m.
\end{aligned}$$

Lemma 5. *Let $z = x_1^{e_1} x_2^{e_2} \dots x_m^{e_m}$. Let $d = \sum_{i=1}^m e_i(m-i)$, $e = \sum_{i=1}^m e_i = \deg(z)$ and assume $d < p$. Then*

$$\mathrm{wt}(z) = d + 1.$$

Proof. Applying Proposition 2 repeatedly and noting that $\mathrm{wt}(x_i) = m - i + 1$, we find

$$\begin{aligned}
\mathrm{wt}(z) &= \sum_{i=1}^{m} (e_i(m-i+1) - e_i + 1) - (n-1)\\
&= \sum_{i=1}^{m} (e_i(m-i)) + 1 = d + 1.
\end{aligned}$$

□

3. Hilbert series

Let $\Bbbk$ be a field and let $S = \oplus_{i \geq 0} S_i$ be a positively graded $\Bbbk$-vector space. The dimension of each graded component of S is encoded in its Hilbert Series

$$H(S,t) = \sum_{i \geq 0} \dim(S_i) t^i.$$

Proposition 4 implies that $H(\Bbbk[V,W]^G, t) = H(K_n, t)$. In this section we will outline a method for computing $H(K_n, t)$.

Each homogeneous component $\Bbbk[V]_i$ of $\Bbbk[V]$ is a $\Bbbk G$-module. As such, each one decomposes as a direct sum of modules isomorphic to V_k for some values of k. Write $\mu_k(\Bbbk[V]_i)$ for the multiplicity of V_k in $\Bbbk[V]_i$ and define the series

$$H_k(\Bbbk[V]) = \sum_{i \geq 0} \mu_k(\Bbbk[V]_i) t^i.$$

The series $H_k(\Bbbk[V_m])$ were studied by Hughes and Kemper in [9]. They can also be used to compute the Hilbert series of $\Bbbk[V_m]^G$; since $\dim(V_k^G) = 1$ for all $k = 1, \dots, p$ we have

$$H(\Bbbk[V_m]^G, t) = \sum_{k=1}^{p} H_k(\Bbbk[V_m], t). \tag{7}$$

Now observe that

$$\dim(\ker(\Delta^n|_{V_k})) = \begin{cases} n & k \geq n \\ k & \text{otherwise.} \end{cases}$$

Therefore

$$H(K_n, t) = \sum_{k=1}^{n-1} k H_k(\Bbbk[V], t) + \sum_{k=n}^{p} n H_k(\Bbbk[V], t).$$

We can write this as as a series not depending on p:

$$H(K_n, t) = nH(\Bbbk[V]^G, t) - (\sum_{k=1}^{n-1}(n-k)H_k(\Bbbk[V], t)). \tag{8}$$

using equation (7).

We will need the Hilbert Series of $I_n^G = \Bbbk[V]^G \cap I_n$ in the final section. For all $k = 1, \ldots, p$ we have

$$\dim(\Delta^n(V_k))^G = \begin{cases} 1 & k > n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$H(I_n^G, t) = \sum_{k=n+1}^{p} H_k(\Bbbk[V], t),$$

which we can write independently of p as

$$H(I_n^G, t) = H(\Bbbk[V]^G, t) - (\sum_{k=1}^{n} H_k(\Bbbk[V], t)). \tag{9}$$

4. Decomposition Theorems

In this section we will compute the series $H_k(\Bbbk[V_2], t)$ and $H_k(\Bbbk[V_3], t)$ for all $k = 1, \ldots, p-1$.

Hughes and Kemper [9, Theorem 3.4] give the formula

$$H_k(\Bbbk[V_m], t) = \sum_{\gamma \in M_{2p}} \frac{\gamma - \gamma^{-1}}{2p} \gamma^{-k} \frac{1 - \gamma^{p(m-1)} t^p}{1 - t^p} \prod_{j=0}^{m-1} (1 - \gamma^{m-1-2j} t)^{-1}, \tag{10}$$

where M_{2p} represents the set of $2p$th roots of unity in $\mathbb{C}$. A similar formula is given for $H_p(\Bbbk[V], t)$ but we will not need this. The following result can be derived from the formula above, but follows more easily from [4, Proposition 3.4]:

Lemma 6. $H_k(\Bbbk[V_2, t]) = \frac{t^{k-1}}{1-t^p}$.

For V_3 we will have to use Equation (10). This becomes

$$H_k(\Bbbk[V_3], t) = \frac{1}{2p(1-t)} \sum_{\gamma \in M_{2p}} \frac{(\gamma - \gamma^{-1})\gamma^{-k+2}}{(1-\gamma^2 t)(\gamma^2 - t)}.$$

Lemma 7.

$$H_k(\Bbbk[V_3], t) = \begin{cases} \frac{t^{p-l} - t^{p-l-1} + t^{l+1} - t^l}{(1-t)(1-t^2)(1-t^p)} & \textit{if } k = 2l+1 \textit{ is odd} \\ 0 & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. We evaluate

$$\frac{(\gamma - \gamma^{-1})\gamma^{-k+2}}{(1-\gamma^2 t)(\gamma^2 - t)} = \frac{A}{\gamma - t^{\frac{1}{2}}} + \frac{B}{\gamma + t^{\frac{1}{2}}} + \frac{C}{1 - \gamma t^{\frac{1}{2}}} + \frac{D}{1 + \gamma t^{\frac{1}{2}}}$$

using partial fractions, finding

$$A = \frac{t^{-l+1} - t^{-l}}{(2t^{\frac{1}{2}})(1-t^2)},$$

$$B = (-1)^{-k+3}\frac{t^{-l+1} - t^{-l}}{(-2t^{\frac{1}{2}})(1-t^2)},$$

$$C = \frac{t^{l-1} - t^{l}}{2(t^{-1} - t)},$$

$$D = (-1)^{-k+3}\frac{t^{l-1} - t^{l}}{2(t^{-1} - t)}.$$

Now we compute:

$$\begin{aligned}
\sum_{\gamma \in M_{2p}} \frac{1}{\gamma - t^{\frac{1}{2}}} &= \sum_{\gamma \in M_{2p}} \frac{-t^{-\frac{1}{2}}}{1 - \gamma t^{-\frac{1}{2}}} \\
&= -t^{-\frac{1}{2}} \sum_{i=0}^{\infty} \sum_{\gamma \in M_{2p}} (\gamma t^{-\frac{1}{2}})^i \\
&= -t^{-\frac{1}{2}} 2p \sum_{i=0}^{\infty} (t^{-\frac{1}{2}})^{2pi} \\
&= -t^{-\frac{1}{2}} 2p \frac{1}{1 - (t^{-\frac{1}{2}})^{2p}} \\
&= -t^{\frac{1}{2}} 2p \frac{1}{1 - t^{-p}} \\
&= 2p \frac{t^{p-\frac{1}{2}}}{1 - t^p}
\end{aligned}$$

Similarly we have

$$\sum_{\gamma \in M_{2p}} \frac{1}{\gamma + t^{\frac{1}{2}}} = -2p \frac{t^{p-\frac{1}{2}}}{1 - t^p}$$

while

$$\begin{aligned}
\sum_{\gamma \in M_{2p}} \frac{1}{1 - \gamma t^{\frac{1}{2}}} &= \sum_{i=0}^{\infty} \sum_{\gamma \in M_{2p}} (\gamma t^{\frac{1}{2}})^i \\
&= 2p \sum_{i=0}^{\infty} (t^{\frac{1}{2}})^{2pi} \\
&= 2p \sum_{i=0}^{\infty} (t^{pi}) \\
&= 2p \frac{1}{1 - t^p}
\end{aligned}$$

and similarly

$$\sum_{\gamma \in M_{2p}} \frac{1}{1 + \gamma t^{\frac{1}{2}}} = 2p \frac{1}{1 - t^p}$$

as $\{-\gamma : \gamma \in M_{2p}\} = M_{2p}$.

It follows that

$$\begin{aligned} H_k(\Bbbk[V_3],t) &= \frac{1}{2p(1-t)}\left(\frac{(A-B)2pt^{p-\frac{1}{2}}}{1-t^p}+\frac{2p(C+D)}{1-t^p}\right)\\ &= \frac{1}{(1-t)(1-t^p)}\left(\frac{(1+(-1)^{-k+3})(t^{p-l}-t^{p-l-1})}{2(1-t^2)}+\frac{(1+(-1)^{-k+3})(t^{l-1}-t^l)}{2(t^{-1}-t)}\right)\\ &= \begin{cases} \frac{t^{p-l}-t^{p-l-1}+t^{l+1}-t^l}{(1-t)(1-t^2)(1-t^p)} & \text{if } k \text{ is odd}\\ 0 & \text{if } k \text{ is even}\end{cases} \end{aligned}$$

as required. □

5. Main results: V_2

We are now in a position to state our main results. First, suppose $V = V_2$ and $W = V_n$ where $n \leq p$. Then it's well known that $\Bbbk[V]^G$ is a polynomial ring, generated by x_2 and

$$N = \prod_{i=0}^{p-1} \sigma^i(x_1) = x_1^p - x_1x_2^{p-1}.$$

Therefore we have

$$H(\Bbbk[V]^G,t) = \frac{1}{(1-t)(1-t^p)}. \tag{11}$$

Proposition 8. *We have*

$$H(K_n,t) = H(\Bbbk[V,W]^G,t) = \frac{1+t+t^2+\ldots+t^{n-1}}{(1-t)(1-t^p)}.$$

Proof. Using equations (8) and (11) and Lemma 6 we have

$$H(K_n,t) = \frac{n}{(1-t)(1-t^p)} - \sum_{k=1}^{n-1}\frac{(n-k)t^{k-1}}{1-t^p} = \frac{1+t+t^2+\ldots+t^{n-1}}{(1-t)(1-t^p)}.$$

The result now follows from Proposition 4. □

Theorem 9. *The module of covariants $\Bbbk[V,W]^G$ is generated freely over $\Bbbk[V]^G$ by*

$$\{\Theta(x_1^k) : k = 0,\ldots,n-1\}.$$

where $\Theta(x_1^0) = \Theta(1) = w_1$.

Note that, by Proposition 1(i), $\Bbbk[V,W]^G$ is free over $\Bbbk[V]^G$ and we could use [1, Theorem 3] to check our proposed module generators. However, we prefer a more direct approach.

Proof. It follows from Lemma 5 that $\mathrm{wt}(x_1^k) = k+1$. Therefore $\mathrm{Supp}(\Theta(x_1^k)) = k+1$, and so it's clear that the $\Bbbk[V]^G$-submodule M of $\Bbbk[V,W]^G$ generated by the proposed generating set is free. Moreover, as $\deg(\Theta(x_1^k)) = k$, M has Hilbert series

$$\frac{1+t+t^2+\ldots+t^{n-1}}{(1-t)(1-t^p)}.$$

But by Proposition 8, this is the Hilbert series of $\Bbbk[V,W]^G$. Therefore $M = \Bbbk[V,W]^G$ as required. □

Corollary 10. *K_n is a free $\Bbbk[V^G]$-module, generated by $\{x_1^k : k = 0,\ldots,n-1\}$.*

Proof. Follows from Theorem 9 above and the proof of Proposition 4. □

Remark 11. The above was also obtained, in the special case $n = p-1$, by Erkuş and Madran [5].

6. Main results: V_3

In this section let p be an odd prime and $V = V_3$. We begin by describing $\Bbbk[V]^G$. This has been done in several places before, for example [3] and [10, Theorem 5.8], but we include this for completeness.

We use a graded reverse lexicographic order on monomials $\Bbbk[V]$ with $x_1 > x_2 > x_3$. If $f \in \Bbbk[V]$ then the *lead term* of f is the term with the largest monomial in our order and the *lead monomial* is the corresponding monomial. If $f, g \in \Bbbk[V]$ we will write

$$f > g$$

if the lead monomial of f is greater than the lead monomial of g.

The results of section 3 can be used to show

$$H(\Bbbk[V]^G, t) = \frac{1 + t^p}{(1-t)(1-t^2)(1-t^p)}. \tag{12}$$

Note that using the given order, we have

$$f > \Delta(f)$$

for all $f \in \Bbbk[V]$.

We recall two popular means of constructing invariants. Let $f \in \Bbbk[V]$. As mentioned in section 2, the transfer

$$\Delta^{p-1}(f) = \mathrm{Tr}^G(f) = \sum_{i=0}^{p-1} (\sigma^i f)$$

and also the norm

$$N(f) = \prod_{i=0}^{p-1} (\sigma^i f)$$

of f both lie in $\Bbbk[V]^G$. It is easily shown that

$$\begin{aligned} a_1 &:= x_3, \\ a_2 &:= x_2^2 - 2x_1x_3 - x_2x_3, \\ a_3 &:= N(x_1) = \prod_{i=0}^{p-1} \sigma^i(x_1) \end{aligned}$$

are invariants, and looking at their lead terms tells us that they form a homogeneous system of parameters for $\Bbbk[V]^G$, with degrees $1, 2$ and p.

Proposition 12. *Let $f \in \Bbbk[V]^G$ be any invariant with lead term x_2^p. Let $A = \Bbbk[a_1, a_2, a_3]$. Then $f \notin A$. Consequently $\Bbbk[V]^G$ is a free A-module, whose generators are 1 and f.*

Proof. It is clear that $f \notin A$, as its lead term is not in the subalgebra of $\Bbbk[V]$ generated by the lead terms of a_1, a_2 and a_3. Therefore the A-submodule of $\Bbbk[V]^G$ generated by 1 and f has Hilbert series

$$\frac{1 + t^p}{(1-t)(1-t^2)(1-t^p)}$$

which is the Hilbert series of $\Bbbk[V]^G$ as required. □

The obvious choice of invariant with lead term x_2^p is $N(x_2)$. However, we will use $\mathrm{Tr}^G(x_1^{p-1}x_2)$ instead. For the calculation of the lead term of this invariant see [11, Lemma 3.1] or Lemma 16 to come.

The following observation is a consequence of the generating set above.

Lemma 13. *Let $f \in A$. Then the lead term of f is of the form $x_1^{pi}x_2^{2j}x_3^k$ for some positive integers i, j, k.*

Now let $W = V_n$ for some $n \leq p$. For the rest of this section, we set $l = \frac{1}{2}n$ if n is even, with $l = \frac{1}{2}(n-1)$ if n is odd. Our first task is to compute the Hilbert Series of $\Bbbk[V,W]^G$. Once more we use equation (8) and the bijection Θ to do this. We omit the details.

Proposition 14.

$$H(\Bbbk[V,W]^G, t) = \frac{1 + 2t + 2t^2 + \ldots + 2t^l + 2t^{p-l} + 2t^{p-l+1} + \ldots + t^p}{(1-t)(1-t^2)(1-t^p)}$$

if n is odd, while

$$H(\Bbbk[V,W]^G, t) = \frac{1 + 2t + 2t^2 + \ldots + 2t^{l-1} + t^l + t^{p-l} + 2t^{p-l+1} + \ldots + 2t^{p-1} + t^p}{(1-t)(1-t^2)(1-t^p)}$$

if n is even.

Next, we need some information about the lead monomials of certain polynomials:

Lemma 15. *Let $j \leq k < p$. Then $\Delta^j(x_1^k)$ has lead term*

$$\frac{k!}{(k-j)!}x_1^{k-j}x_2^j.$$

Proof. The proof is by induction on j, the case $j = 0$ being clear. Suppose $1 \leq j < k$ and

$$\Delta^j(x_1^k) = \frac{k!}{(k-j)!}x_1^{k-j}x_2^j + g$$

where $g \in \Bbbk[V]$ has lead monomial $\leq x_1^{k-j-1}x_2^{j+1}$. Then

$$\begin{aligned}\Delta^{j+1}(x_1^k) &= \frac{k!}{(k-j)!}\Delta(x_1^{k-j}x_2^j) + \Delta(g)\\ &= \frac{k!}{(k-j)!}\Delta(x_1^{k-j})\sigma(x_2^j) + x_1^{k-j}\Delta(x_2^j) + \Delta(g).\end{aligned}$$

Note that the lead monomial of $\Delta(g)$ is $< x_1^{k-j-1}x_2^{j+1}$. Now applying (3) shows that $\Delta(x_2^j)$ is divisible by x_3 and

$$\begin{aligned}\Delta(x_1^{k-j}) &= x_2(x_1^{k-j-1} + x_1^{k-j-2}\sigma(x_1) + \ldots + \sigma(x_1)^{k-j-1})\\ &= (k-j)x_1^{k-j-1}x_2 + \text{ smaller terms.}\end{aligned}$$

In addition,

$$\sigma(x_2^j) = (x_2 + x_3)^j = x_2^j + \text{ smaller terms.}$$

Therefore the lead term of $\Delta^{j+1}(x_1^k)$ is

$$(k-j)\frac{k!}{(k-j)!}x_1^{k-j-1}x_2^{j+1} = \frac{k!}{(k-j-1)!}x_1^{k-j-1}x_2^{j+1}$$

as required.

□

Similarly we have

Lemma 16. *Let $j \leq k < p$. Then $\Delta^j(x_1^k x_2)$ has lead term*

$$\frac{k!}{(k-j)!}x_1^{k-j}x_2^{j+1}.$$

Proof. We have by (2)

$$\Delta^j(x_1^k x_2) = \sum_{i=0}^{j} \binom{j}{i} \Delta^{j-i}(x_1^k)\sigma^i(\Delta^i(x_2)).$$

Only the first two terms are nonzero, hence

$$\begin{aligned}\Delta^j(x_1^k x_2) &= \Delta^j(x_1^k)x_2 + j\Delta^{j-1}(x_1^k)x_3. \\ &= \frac{k!}{(k-j)!}x_1^{k-j}x_2^{j+1} + \text{ smaller terms}\end{aligned}$$

where we used Lemma 15 is the last step. □

We are now ready to state our main results. Let $V = V_3$ and $W = V_n$. For any $i = 0, 1, \dots, n-1$ we define monomials

$$M_i = \begin{cases} x_1^{i/2} & \text{if } i \text{ is even,} \\ x_1^{(i-1)/2}x_2 & \text{if } i \text{ is odd.} \end{cases}$$

and polynomials

$$P_i = \begin{cases} \Delta(x_1^{p-i/2}) & \text{if } i \text{ is even, } i > 0, \\ x_1^{p-(i+1)/2} & \text{if } i \text{ is odd.} \end{cases}$$

with $P_0 = x_1^{p-1}x_2$.

Theorem 17. *Let $n \le p$. Then K_n is a free A-module, generated by*

$$S_n = \{M_0, M_1, \dots, M_{n-1}, \Delta^{p-n}(P_0), \Delta^{p-n}(P_1), \dots, \Delta^{p-n}(P_{n-1})\}.$$

Proof. By Lemma 2, the weight of M_i is $i+1$ for $i < p$, while the weight of P_i is

$$\begin{cases} p & i \text{ odd or zero} \\ p-1 & i \text{ even, } i > 0. \end{cases}$$

Therefore the given polynomials all lie in K_n. Further, the degree of M_i is $\lceil \frac{i}{2} \rceil$ and the degree of P_i is $p - \lceil \frac{i}{2} \rceil$ which shows that the A-module generated by S_n has Hilbert series bounded above by the Hilbert series of K_n given in Proposition 14, with equality if and only if it is free. Therefore it is enough to prove that S_n is linearly independent over A.

Applying Lemmas 15 and 16, the lead monomials of S_n are

$$\{1, x_2, x_1, x_1x_2, \dots, x_1^{l-1}x_2, x_1^l,$$
$$x_1^{n-l-1}x_2^{p-n+1}, x_1^{n-l}x_2^{p-n}, \dots, x_1^{n-2}x_2^{p-n+1}, x_1^{n-1}x_2^{p-n}, x_1^{n-1}x_2^{p-n+1}\}$$

if n is odd, and

$$\{1, x_2, x_1, x_1x_2, \dots, x_1^{l-2}x_2, x_1^{l-1}, x_1^{l-1}x_2,$$
$$x_1^{n-l}x_2^{p-n}, x_1^{n-l}x_2^{p-n+1}, x_1^{n-l+1}x_2^{p-n} \dots, x_1^{n-2}x_2^{p-n+1}, x_1^{n-1}x_2^{p-n}, x_1^{n-1}x_2^{p-n+1}\}$$

if n is even.

In either case, we note that none of the claimed generators have lead term divisible by x_3, that each has x_1-degree $< p$, that there are at most two elements in S_n with the same x_1-degree, and that when this happens these elements have x_2-degrees differing by 1. Combined with Lemma 13, we see that for every possible choice of $f \in A$ and $g \in S_n$, the lead monomial of fg is different. Therefore there cannot be any A-linear relations between the elements of S_n. □

Remark 18. A generating set for K_{p-1} over a different system of parameters can be found in [5].

Corollary 19. *Let $n \leq p$. Then $\Bbbk[V,W]^G$ is a Cohen-Macaulay module, generated over A by*

$$\{\Theta(M_0), \Theta(M_1), \dots, \Theta(M_{n-1}), \Theta(P_0), \Theta(\Delta^{p-n}(P_1)), \dots, \Theta(\Delta^{p-n}(P_{n-1}))\}.$$

Proof. Follows from Theorem 17 and the proof of Proposition 4. □

7. Application to transfers

The transfer ideal $\mathrm{Tr}^G(\Bbbk[V])$ is widely studied in invariant theory. In the notation of this article, we have $\mathrm{Tr}^G(\Bbbk[V]) = I_{p-1}^G = I_{p-1}$. In this section, we use our work on covariants to give minimal $\Bbbk[V]^G$-generating sets of the the ideals I_{n-1}^G for each $n = 1, 2, \dots, p$ when $V = V_2$, and minimal A-generating sets of the the ideals I_{n-1}^G for each $n = 1, 2, \dots, p$ when $V = V_3$. We retain the notation of sections 5 and 6.

Theorem 20. *Let $V = V_2$ and $1 \leq n \leq p$. Then I_{n-1}^G is a free $\Bbbk[V]^G$-module, generated by x_2^{n-1}.*

Proof. The same argument as in Lemma 15 implies that $\Delta^{n-1}(x_1^{n-1}) = \lambda x_2^{n-1}$ for some nonzero constant λ, so $x_2^{n-1} \in I_{n-1}^G$. Using (9) we see that

$$H(I_{n-1}^G, t) = \frac{t^{n-1}}{(1-t)(1-t^n)}.$$

As this is the Hilbert series of the ideal $x_2^{n-1}\Bbbk[V]^G$, the result follows. □

For $V = V_3$ we need to do a bit more work. We define a set of invariants

$$T_{n-1} = \{\Delta^{n-1}(M_{n-1})\} \cup \{\Delta^{p-1}(P_i) : i \text{ odd or zero}, i < n\}.$$

Bearing in mind the weight of M_{n-1} is n, and the weight of each P_i above is p, it's clear that $T_{n-1} \subset I_{n-1}^G$. We claim that

Proposition 21. *T_{n-1} generates I_{n-1}^G as an A-module.*

Proof. Let $h \in I_{n-1}^G$. Then we can write $h = \Delta^{n-1}(f)$ for some $f \in \Bbbk[V]^G$ with weight n, and by Proposition 3 we have $\Theta(f) \in \Bbbk[V, V_n]^G$. By Corollary 19 we can find elements $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \beta_0, \beta_1, \dots, \beta_{n-1} \in A$ such that

$$\Theta(f) = \sum_{i=0}^{n-1} \alpha_i \Theta(M_i) + \sum_{i=0}^{n-1} \beta_i \Theta(\Delta^{p-n}(P_i)).$$

Equating coefficients of w_n in the above we obtain

$$h = \sum_{i=0}^{n-1} \alpha_i \Delta^{n-1}(M_i) + \sum_{i=0}^{n-1} \beta_i \Delta^{p-1}(P_i))$$

but since $\Delta^{n-1}(M_i) = 0$ for $i < n-1$ and $\Delta^{p-1}(P_i) = 0$ when i is even and $i > 0$, we get $h \in AT_n$ as desired. □

T_{n-1} does not generate I_{n-1}^G freely over A. To see this, note that if T_{n-1} were free over A, the resulting module would have Hilbert series

$$\frac{t^l + t^{p-l} + t^{p-l+1} + \ldots + t^p}{(1-t)(1-t^2)(1-t^p)}.$$

But using (9) to calculate the Hilbert series of I_n^G yields

$$H(I_{n-1}^G, t) = \frac{t^l + t^{p-l}}{(1-t)(1-t^2)(1-t^p)} \tag{13}$$

which is strictly smaller. We claim, however, that T_n is a minimal generating set. The first step in our argument requires more knowledge of certain lead monomials:

Lemma 22. *Let $j \leq k$ with $j+k<p$. Then $\Delta^{k+j}(x_1^k)$ can be expressed as*

$$2^{-j}(j+k)!\binom{k}{j}x_2^{k-j}x_3^j+\mu_{j,k}x_1x_2^{k-j-2}x_3^{j+1}+\text{ smaller terms}$$

for some constant $\mu_{j,k}\in\Bbbk$, where $\mu_{j,k}=0$ if $j-k<2$. In particular, the lead monomial of $\Delta^{k+j}(x_1^k)$ is $x_2^{k-j}x_3^j$.

Proof. For shorthand we write

$$\lambda_{j,k}=2^{-j}(j+k)!\binom{k}{j}.$$

We begin by showing, for all $0<j\leq k$, that

$$\lambda_{j,k+1}=(j+k+1)\lambda_{j,k}+\binom{j+k+1}{2}\lambda_{j-1,k}. \tag{14}$$

The author wishes to thank Fedor Petrov for pointing out this fact. To prove it, note that

$$\binom{j+k+1}{2}\lambda_{j-1,k}+(j+k+1)\lambda_{j,k}$$

$$\begin{aligned}
&=\frac{(j+k+1)(j+k)}{2}2^{-j+1}(j+k-1)!\binom{k}{j-1}+(j+k+1)2^{-j}(j+k)!\binom{k}{j}\\
&=2^{-j}(j+k+1)!\left(\binom{k}{j-1}+\binom{k}{j}\right)\\
&=2^{-j}(j+k+1)!\binom{k+1}{j}\\
&=\lambda_{j,k+1}
\end{aligned}$$

as required.

The proof is by induction on j. First suppose $j=0$. We must show that

$$\Delta^k(x_1^k)=k!x_2^k+\mu_{0,k}x_1x_2^{k-2}x_3+\text{ smaller terms.} \tag{15}$$

We prove this by induction on k. The case $k=1$ is clear (with $\mu_{0,1}=0$), so let $k\geq 1$. Then we have

$$\begin{aligned}
\Delta^{k+1}(x_1^{k+1})&=\Delta^{k+1}(x_1^k\cdot x_1)\\
&=\sum_{i=0}^{k+1}\binom{k+1}{i}\Delta^{k+1-i}(x_1^k)\sigma^i(\Delta^i(x_1))\\
&=x_1\Delta^{k+1}(x_1^k)+(k+1)(x_2+x_3)\Delta^k(x_1^k)+\binom{k+1}{2}x_3\Delta^{k-1}(x_1^k).
\end{aligned}$$

Now by Lemma 15 we have

$$\Delta^{k-1}(x_1^k)=k!x_1x_2^{k-1}+f$$

for some $f\in\Bbbk[V]$ with lead monomial $\leq x_2^k$. By induction we have

$$\Delta^k(x_1^k)=k!x_2^k+\mu_{0,k}x_1x_2^{k-2}x_3+\text{ smaller terms}$$

and

$$\Delta^{k+1}x_1^k=k!\Delta(x_2^k)+\mu_{0,k}x_3\Delta(x_1x_2^{k-2})+\text{smaller terms.}$$

$$=k!x_3(x_2^{k-1}+x_2^{k-2}\sigma(x_2)+\ldots+\sigma(x_2)^{k-1})+\mu_{0,k}x_3(x_2\sigma(x_2^{k-2})+x_1\Delta(x^{k-2}))+\text{smaller terms}$$

$$=(k.k!+\mu_{0,k})x_2^{k-1}x_3+\text{ smaller terms.}$$

So, ignoring terms smaller than $x_1x_2^{k-1}x_3$ we have

$$\Delta^{k+1}(x_1^{k+1}) = (k.k! + \mu_{0,k})x_1x_2^{k-1}x_3 + (k+1)!x_2^{k+1} + (k+1)\mu_{0,k}x_1x_2^{k-1}x_3 + k!\binom{k+1}{2}x_1x_2^{k-1}x_3$$
$$= (k+1)!x_2^{k+1} + (k!(k+\binom{k+1}{2})) + (k+2)\mu_{0,k})x_1x_2^{k-1}x_3$$

from which the claim (15) follows.

Now suppose $j > 0$. We proceed by induction on k. The initial case is $k = j$, so we must first show that

$$\Delta^{2k}(x_1^k) = 2^{-k}(2k)!x_3^k.$$

We prove this by induction on k. The result is clear when $k = 1$. Suppose that $k \geq 1$, then we have by (2)

$$\Delta^{2k+2}(x_1^{k+1}) = x_1\Delta^{2k+2}(x_1^k)+(2k+2)(x_2+x_3)\Delta^{2k+1}(x_1^k)+\frac{(2k+2)(2k+1)}{2}x_3\Delta^{2k}(x_1^k).$$

But by Lemma 5, the weight of x_1^k is $2k+1$, so the first two terms vanish. By induction we are left with

$$\Delta^{2k+2}(x_1^{k+1}) = \frac{(2k+2)(2k+1)}{2}x_3\frac{(2k)!}{2^k}x_3^k = \frac{(2k+2)!}{2^{k+1}}x_3^{k+1}$$

as required.

Now suppose $k \geq j$, then we have

$$\Delta^{j+k+1}(x_1^{k+1}) = \Delta^{j+k+1}(x_1^k \cdot x_1)$$
$$= \sum_{i=0}^{j+k+1}\binom{j+k+1}{i}\Delta^{j+k+1-i}(x_1^k)\sigma^i(\Delta^i(x_1))$$
$$= x_1\Delta^{j+k+1}(x_1^k) + (j+k+1)(x_2+x_3)\Delta^{j+k}(x_1^k) + \binom{j+k+1}{2}x_3\Delta^{j-1+k}(x_1^k).$$

Now by induction on k we have we have

$$\Delta^{j+k}(x_1^k) = \lambda_{j,k}x_2^{k-j}x_3^j + \mu_{j,k}x_1x_2^{k-j-2}x_3^{j+1} + \text{ smaller terms.}$$

So

$$\Delta^{j+k+1}(x_1^k) = \lambda_{j,k}x_3^j\Delta(x_2^{k-j}) + \mu_{j,k}x_3^{j+1}\Delta(x_1x_2^{k-j-2}) + \text{ smaller terms}$$

$$= \lambda_{j,k}x_3^j(x_3)(x_2^{k-j-1} + x_2^{k-j-2}\sigma(x_2) + \ldots + \sigma(x_2)^{k-j-1})$$
$$+ \mu_{j,k}x_3^{j+1}(x_2\sigma(x_2^{k-j-2}) + x_1\Delta(x_2^{k-j-2})) + \text{ smaller terms}$$

$$= (\lambda_{j,k}(k-j) + \mu_{j,k})x_3^{j+1}x_2^{k-j-2} + \text{ smaller terms.}$$

Also by induction on j we have

$$\Delta^{j-1+k}(x_1^k) = \lambda_{j-1,k}x_2^{k-j+1}x_3^{j-1} + \mu_{j-1,k}x_1x_2^{k-j-1}x_3^j + \text{ smaller terms.}$$

So, ignoring terms smaller than $x_1x_2^{k-j-1}x_3^{j+1}$ we have

$$\begin{aligned}\Delta^{j+k+1}(x_1^{k+1}) &= (\lambda_{j,k}(k-j)+\mu_{j,k})x_1x_3^{j+1}x_2^{k-j-2}\\ &+(j+k+1)(\lambda_{j,k}x_2^{k+1-j}x_3^j+\mu_{j,k}x_1x_2^{k-j-1}x_3^{j+1})\\ &+\binom{j+k+1}{2}(\lambda_{j-1,k}x_2^{k-j+1}x_3^j+\mu_{j-1,k}x_1x_2^{k-j-1}x_3^{j+1})\end{aligned}$$

$$\begin{aligned}&=\left((j+k+1)\lambda_{j,k}+\binom{j+k+1}{2}\lambda_{j-1,k}\right)x_2^{k+1-j}x_3^j\\ &+(\lambda_{j,k}(k-j)+(j+k+2)\mu_{j,k}+\binom{j+k+1}{2}\mu_{j-1,k})x_1x_2^{k-j-1}x_3^{j+1}\end{aligned}$$

$$\begin{aligned}&=\lambda_{j,k+1}x_2^{k+1-j}x_3^j+\\ &(\lambda_{j,k}(k-j)+(j+k+2)\mu_{j,k}+\binom{j+k+1}{2}\mu_{j-1,k})x_1x_2^{k-j-1}x_3^{j+1}\end{aligned}$$

where we used the observation at the beginning of the proof in the final step.

This completes the proof of the formula for $\Delta^{j+k}(x_1^k)$. Finally, note that $\lambda_{j,k}\neq 0$ modulo p if $j+k<p$. □

We can use this result, along with Lemma 16 to determine the lead monomial of each element of T_{n-1}: we have

- $LM(\Delta^{n-1}M_{n-1})=x_3^l$;
- $LM(\Delta^{p-1}(P_0))=x_2^p$;
- $LM(\Delta^{p-1}(P_i))=x_2^{p-i}x_3^{(i-1)/2}$ when i is odd.

In particular for each $i<n$ odd or $i=0$ we have that

$$\Delta^{p-1}(P_i)\notin A(\Delta^{n-1}(M_{n-1}),\Delta^{p-1}(P_j): j>i, \text{j odd}),$$

which is the the ideal generated by the elements of T_{n-1} with degree smaller than the degree of $\Delta^{p-1}(P_i)$, since each of these had lead monomial divisible by a larger power of x_3 than $(i-1)/2$. This shows that T_{n-1} is indeed a minimal generating set.

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