MS03120 Real and Complex Analysis

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2016–2017

Part II

Measure Theory

Chapter 6

Introduction

6.1 Deficiencies in Riemann Integration

Recall the definition of Riemann integration.

- A partition P of an interval $[a, b]$ is a sequence of finitely many numbers $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.
- Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. The upper and lower Riemann sums of f with respect to P are

$$U(f; P) = \sum_{k=1}^{n} (x_k - x_{k-1}) \sup_{x \in [x_{k-1}, x_k]} f(x)$$
$$\text{and} \qquad L(f; P) = \sum_{k=1}^{n} (x_k - x_{k-1}) \inf_{x \in [x_{k-1}, x_k]} f(x)$$

  respectively.
- A bounded function is Riemann integrable if its smallest upper Riemann sum is equal to its largest lower Riemann sum

Definition 6.1: Riemann integration

A bounded function $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable if

$$\inf\{U(f; P) \,|\, P \text{ is a partition of } [a, b]\} = \sup\{L(f; P) \,|\, P \text{ is a partition of } [a, b]\} \tag{6.1}$$

- The Riemann integral of a bounded function $f\colon [a, b] \to \mathbb{R}$ is defined as the common value of (6.1) i.e.

$$\int_a^b f(x)\, \mathrm{d}x := \inf\{U(f; P) \,|\, P \text{ is a partition of } [a, b]\} = \sup\{L(f; P) \,|\, P \text{ is a partition of } [a, b]\}$$

If $f\colon [a, b] \to \mathbb{R}$ is continuous then f is Riemann integrable.

There are some flaws with Riemann integration.

1. There are 'nice' functions that are not Riemann integrable.
2. If f_n is a sequence of Riemann integrable functions and $f_n \to f$, then f may not be Riemann integrable.
3. Riemann Integration cannot be applied directly to unbounded domains such as the real line $\mathbb{R}$. Instead the integral is defined by taking further limits.
4. Riemann integration requires the ordering of the real line. We can't use it to define integration on more general spaces.

✎ **Example 6.2.**
Define the function $f: [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This is the **indicator** function of the set of rational numbers $\mathbb{Q}$. (We will use indicator functions frequently).

Note that f is bounded, so the upper and lower Riemann sums are defined for every partition P.

✎ **Example 6.3.**
Order the rationals in $[0,1]$ so $\mathbb{Q} \cap [0,1] = \{r_k\}_{k=1}^{\infty}$.

For each $n \in \mathbb{N}$ define the function

$$f_n(x) = \begin{cases} 1 & x = r_k \text{ for some } k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

1. f_n is Riemann integrable

2. f_n converges pointwise to the indicator function of $\mathbb{Q}$.

Also note that $0 \leq f_n(x) \leq f_{n+1}(x)$ for all $x \in [0,1]$, which means that the functions f_n converge monotonically. This is a very 'nice' convergence, but nevertheless the limit is **not** Riemann Integrable.

6.1.1 A first look at Lebesgue integration

6.2 Naive Measure Theory

Essential to a new theory of integration is the ability to measure the 'size' of sets $A \subset \mathbb{R}$.

Our ultimate goal is to establish a mathematical 'measure theory' which puts this on a rigorous, formal footing.

However, this is not just a mathematical 'box ticking' exercise to make sure that our intuitions can be written down mathematically. Basically, it turns out that our intuition about measuring sets is wrong.

In this chapter we will write down some intuitively reasonable rules for measuring sets. We will then show that these rules lead to massive contradictions: the intuitive rules are inconsistent.

In the next chapter we will replace these intuitively reasonable rules with a more restrictive set of mathematically consistent rules.

Intuitive rules

We would like to define a map μ such that for each subset $A \subset \mathbb{R}$ we assign a value $\mu(A) \in \mathbb{R}^+$ that tells us the 'size' of the set A.

The following axioms for measure seems reasonable.

N1: The unit interval $[0, 1]$ has measure 1, that is

$$\mu([0, 1]) = 1.$$

N2: The union of **countably many disjoint** sets A_j has measure equal to the sum of their measures, that is

$$\mu\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \sum_{j\in\mathbb{N}} \mu(A_j).$$

N3: For each $A \subset \mathbb{R}$ the translated set $A + t = \{a + t \mid a \in A\}$ has the same measure as A, that is

$$\mu(A + t) = \mu(A).$$

Inconsistency

Theorem 6.4:

The axioms $\{N1, N2, N3\}$ are inconsistent.

In the remainder of this section we prove this theorem. To do this we will use one of the axioms of set theory called the axiom of choice:

Axiom of Choice

Let I be a set. Suppose $\{S_i\}_{i\in I}$ is a collection of non-empty sets S_i. Then there exists an collection of elements $\{x_i\}_{i\in I}$ with $x_i \in S_i$ for each $i \in I$.

Constructing a set V

Proposition 6.5:

Consider the group $(\mathbb{R}, +)$. The quotient group $\mathbb{R}/\mathbb{Q}$ has the following properties:

- The elements $\mathbb{Q} + r \in \mathbb{R}/\mathbb{Q}$ are disjoint.
- Each element $\mathbb{Q} + r \in \mathbb{R}/\mathbb{Q}$ is countable.
- $\mathbb{R}/\mathbb{Q}$ is uncountable.
- Each element $\mathbb{Q} + r \in \mathbb{R}/\mathbb{Q}$ intersects $[0, 1]$

✎ Proof.

□

We now use the axiom of choice to define a set V. Consider the collection of sets $\{(\mathbb{Q}+r)\cap[0,1]\}_{r\in\mathbb{R}}$. From the axiom of choice there is a set

$$V := \{v_r\}_{r\in\mathbb{R}} \quad \text{such that} \quad v_r \in (\mathbb{Q}+r)\cap[0,1] \quad \text{for each } r \in \mathbb{R}.$$

The set V is called a **Vitali set**.

Measuring the set V

Let $\{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers in $[-1, 1]$.

Proposition 6.6:

Consider the translated sets $V_k := V + q_k$.

- $V_k \subset [-1, 2]$
- $[0, 1] \subset \bigcup_{k \in \mathbb{N}} V_k$
- The V_k are pairwise disjoint.

✎ Proof.

□

Proposition 6.7: The inconsistency

The measure of V, $\mu(V)$, is not well defined.

✎ Proof.

□

Theorem 6.8: The Banach-Tarski Paradox

Let A be the unit sphere in $\mathbb{R}^3$ and $B = A \cup A + (3, 0, 0)$ (i.e. B is two disjoint copies of the unit sphere). There exists sets $A_1, A_2, \ldots, A_5$ and $B_1, B_2, \ldots, B_5$ such that

- The A_i are pairwise disjoint and $A = A_1 \cup A_2 \cup \ldots \cup A_5$,
- The B_i are pairwise disjoint and $B = B_1 \cup B_2 \cup \ldots \cup B_5$, and
- for each $i = 1, \ldots, 5$ the sets A_i and B_i are isometric.

Chapter 7

Abstract measure theory

Let X be a non-empty set. The set of all subset of X, called the power set of X, is denoted $\mathcal{P}(X)$.

7.1 Sigma algebras

Definition 7.1: Sigma algebra

A collection of sets $\mathcal{A} \subset \mathcal{P}(X)$ is a sigma algebra if

- The empty set $\emptyset \in \mathcal{A}$,
- If $E \in \mathcal{A}$ then the complement $E^c \in \mathcal{A}$,
- If $\{E_j\}_{j=1}^{\infty}$ is a countable collection of sets $E_j \in \mathcal{A}$ then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$.

Theorem 7.2:

Let X be a non-empty set and $\mathcal{A} \subset \mathcal{P}(X)$ be a sigma algebra.

- $X \in \mathcal{A}$.
- If $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$.
- If $\{E_j\}_{j=1}^{\infty}$ is a countable collection of sets $E_j \in \mathcal{A}$ then $\bigcap_{j=1}^{\infty} E_j \in \mathcal{A}$.

✎ Proof.

□

Definition 7.3:

Let X be a non-empty set and $\mathcal{A} \subset \mathcal{P}(X)$ be a sigma algebra. The pair $(X, \mathcal{A})$ is called a **measurable space**.

Generating sigma algebras

Theorem 7.4:

Let X be a non-empty set and let $\mathcal{F} \subset \mathcal{P}(X)$ be a collections of subsets of X. Then there is a smallest sigma algebra that contains $\mathcal{F}$.

Proof.
✎ Let Σ be the collection of sigma algebras of X that contain F.

Σ is non-empty as $\mathcal{P}(X)$ is a sigma algebra and $F \subset \mathcal{P}(X)$, so $\mathcal{P}(X) \in \Sigma$.

Let
$$\sigma(F) = \{E | E \in \mathcal{A} \text{ for all } \mathcal{A} \in \Sigma\}.$$

We now show that $\sigma(F)$ is a sigma algebra

□

Theorem 7.5:

Let $(X, \mathcal{A})$ be a measurable space, Y a set and $f : X \to Y$ a mapping. Then $\{A \subset Y | f^{-1}(A) \in \mathcal{A}\}$ is a sigma algebra.

✎ Proof.

□

7.2 Measures

Definition 7.6:

Let $(X, \mathcal{A})$ be a measurable space. A map $\mu\colon \mathcal{A} \to [0, \infty]$ is a **measure** on $(X, \mathcal{A})$ if

- $\mu(\emptyset) = 0$
- if $\{E_j\}_{j=1}^{\infty}$ is a countable collection of **disjoint** sets $E_j \subset \mathcal{A}$ then

$$\mu\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{i=1}^{\infty} \mu(E_j)$$

Example 7.7.
Let $X = \{1, 2, 3, 4, 5, 6\}$. The following are measures on the measure space $(X, \mathcal{P}(X))$

1. $\mu \equiv 0$ (the trivial measure)
2. μ defined by
$$\mu(E) = \begin{cases} 12 & 2 \in E \\ 0 & \text{otherwise} \end{cases}$$
3. μ defined by $\mu(A) = \operatorname{card}(A)$ (the counting measure)
4. μ defined by $\mu(A) = \operatorname{card}(A)/6$ (the normalised counting measure)
5. μ defined by $\mu(A) = \sum_{x \in A} x$
6. μ defined by $\mu(A) = \sum_{x \in A} x^3$

Definition 7.8:

Let $(X, \mathcal{A})$ be a measurable space and μ be a measure on $(X, \mathcal{A})$.

- The triple $(X, \mathcal{A}, \mu)$ is called a **measure space**.
- If $\mu(X) < \infty$ then μ is called a **finite measure**.
- If $\mu(X) = 1$ then μ is called a **probability measure**.
- If $\{E_j\}_{j=1}^{\infty}$ is a countable collection of sets such that

  a) $X = \bigcup_{j=1}^{\infty} E_j$, and

  b) $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$

  then μ is called a **sigma finite measure**.

Theorem 7.9: Basic properties of measures

Let $(X, \mathcal{A}, \mu)$ be a measure space. The following properties hold

1. Monotonicity: if $E, F \in \mathcal{A}$ and $E \subset F$ then $\mu(E) \leq \mu(F)$
2. Subadditivity: if $\{E_j\}_{j=1}^{\infty}$ is a collection of sets $E_j \in \mathcal{A}$ then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$
3. Continuity from below: if $\{E_j\}_{j=1}^{\infty}$ is a collection of sets $E_j \in \mathcal{A}$ and $E_j \subset E_{j+1}$ for all $j \in \mathbb{N}$ then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

4. Continuity from above: if $\{E_j\}_{j=1}^{\infty}$ is a collection of sets $E_j \in \mathcal{A}$ with $\mu(E_1) < \infty$ and $E_j \supset E_{j+1}$ for all $j \in \mathbb{N}$ then

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

✎ Proof.

□

Chapter 8

Measure Theory on $\mathbb{R}$

The goal of this section is to define a measure on the real line $\mathbb{R}$.

However, our example in Section 6.2 shows that there is **no** measure μ on the measurable space $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ such that

- μ is translation invariant, and
- $\mu([0, 1]) = 1$.

Ultimately we will solve this problem by restricting our attention to the measurable space $(\mathbb{R}, \mathcal{A})$ where $\mathcal{A}$ is some sigma algebra that is smaller than the sigma algebra $\mathcal{P}(\mathbb{R})$.

This means that we will be able to measure all of the sets in $\mathcal{A}$, but there will be some subsets of $\mathbb{R}$ that our measure isn't defined on. These 'non-measurable' sets include the set V from Section 6.2.

The strategy is the following:

1. Define a map $\mu^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, called an outer measure, which is 'almost' a measure.
2. Investigate on which collection of sets $\mathcal{A}$ the map μ^* acts as a measure.
3. Show that this collection of sets $\mathcal{A}$ is a sigma algebra.
4. Restrict μ^* to this collection $\mathcal{A}$, and call this restriction μ.
5. Conclude that the triple $(\mathbb{R}, \mathcal{A}, \mu)$ is a measure space.

8.1 Defining an outer measure

For an interval

$$I = \begin{cases} [a, b]\,, \\ [a, b)\,, \\ (a, b]\,, & or \\ (a, b) \end{cases}$$

let $|I| = b - a$ be the length of the interval. We include the degenerate intervals $[a, a] = a$ and $(a, a) = \emptyset$, which both have length 0.

Definition 8.1: The Lebesgue outer measure

For $A \subset \mathbb{R}$ (i.e. $A \in \mathcal{P}(\mathbb{R})$) let

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} |I_n| \;\middle|\; A \subset \bigcup_{n=1}^{\infty} I_n \quad \{I_n\} \text{ is a countable collection of intervals of } \mathbb{R} \right\}.$$

Theorem 8.2:

The function μ^* satisfies

1. $0 \leq \mu^*(A) \leq \infty$ for all $A \subset \mathbb{R}$
2. $\mu^*(\emptyset) = 0$
3. $\mu^*(A) \leq \mu^*(B)$ for all $A \subset B \subset \mathbb{R}$
4. $\mu^*\left(\bigcup_{m=1}^{\infty} A_m\right) \leq \sum_{m=1}^{\infty} \mu^*(A_m)$ for $A_m \subset \mathbb{R}$.

✎ Proof.

□

Theorem 8.3:

The function μ satisfies

1. If I is an interval then $\mu^*(I) = |I|$
2. $\mu^*(A + h) = \mu^*(A)$ for all $h \in \mathbb{R}$ and all $A \subset \mathbb{R}$.

✎ Proof.

□

Corollary 8.4:

The outer measure μ^* is **not** a measure on the measurable space $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$.

✎ Proof.

□

8.2 Obtaining a measure on $\mathbb{R}$

In order for μ^* to be a measure we countable additivity (i.e. the second property of Definition 7.6), which is that for all countable collections of disjoint sets $\{E_j\}_{j=1}^{\infty}$ we want the equality

$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu^*(E_j). \tag{8.1}$$

- The equality (8.1) **doesn't** hold if we take the E_j to be any disjoint elements of the sigma algebra $\mathcal{P}(\mathbb{R})$.
- The equality (8.1) **does** hold if we take the E_j to be any disjoint elements of the sigma algebra $\{\emptyset, \mathbb{R}\}$.

So what is the biggest sigma-algebra $\mathcal{A}$ for which (8.1) holds?

Definition 8.5: Lebesgue measurable sets

A set $A \subset \mathbb{R}$ is a Lebesgue measurable set iff for all $E \subset \mathbb{R}$

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

The collection of Lebesgue measurable sets is denoted $\mathcal{L}$, i.e.

$$\mathcal{L} = \{A \subset \mathbb{R} \mid A \text{ is a Lebesgue measurable set}\}$$

Lemma 8.6:

1. $\emptyset \in \mathcal{L}$
2. If $A \in \mathcal{L}$ then $A^c \in \mathcal{L}$
3. If $A, B \in \mathcal{L}$ then $A \cup B \in \mathcal{L}$.

✎ Proof.

□

Lemma 8.7:

If $A \in \mathcal{L}$ and $A \cap B = \emptyset$ then

$$\mu^* (A \cup B) = \mu^* (A) + \mu^* (B)$$

✎ Proof.

□

Theorem 8.8:

If $\{A_j\}_{j=1}^{\infty}$ is a collection of sets $A_j \in \mathcal{L}$ then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{L}$.

✎ Proof.

□

Corollary 8.9:

The collection of Lebesgue measurable sets $\mathcal{L}$ is a sigma algebra.

Theorem 8.10:

If $\{E_j\}_{j=1}^{\infty}$ is a disjoint collection of Lebesgue measurable sets then

$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu^*\left(E_j\right).$$

✎ Proof.

□

Definition 8.11: Lebesgue measure

The restriction of μ^* to the collection $\mathcal{L}$ is denoted μ and is called the Lebesgue measure on $\mathbb{R}$.

Corollary 8.12:

The triple $(\mathbb{R}, \mathcal{L}, \mu)$ is a measure space. We call μ the Lebesgue measure on $\mathbb{R}$.

8.2.1 The Lebesgue measuring of some interesting sets

Lemma 8.13:

If $I \subset \mathbb{R}$ is an open interval then $I \in \mathcal{L}$ and $\mu(I) = |I|$.

✎ Proof.

□

Lemma 8.14:

If $C \subset \mathbb{R}$ is a countable set then $C \in \mathcal{L}$ and $\mu(C) = 0$.

✎ Proof.

□

Lemma 8.15:

If C is the Cantor middle third set then $C \in \mathcal{L}$ and $\mu(C) = 0$.

✎ Proof.

□

8.3 Completeness of measures

In a measure space we would like every subset of a measure zero set to be in the sigma algebra.

Definition 8.16:

The measure space $(X, \mathcal{A}, \nu)$ is **complete** if for any $N \in \mathcal{A}$ with $\nu(N) = 0$ then $E \subset N$ implies that $E \in \mathcal{A}$.

Theorem 8.17:

The measure space $(\mathbb{R}, \mathcal{L}, \mu)$ is complete.

✎ Proof.

□

Corollary 8.18:

If C is the Cantor middle third set and $N \subset C$ then $N \in \mathcal{L}$.

8.4 Borel measurable sets

Let $\mathcal{O}(\mathbb{R})$ be the collection of all open subsets of $\mathbb{R}$. The collection $\mathcal{O}(\mathbb{R})$ is not a sigma algebra (see Exercise 1 of Assessment 4). However, we have can create a sigma algebra from $\mathcal{O}(\mathbb{R})$ using Theorem 7.4:

Definition 8.19: The Borel sigma algebra

The Borel sigma algebra is defined by

$$\mathcal{B} = \sigma(\mathcal{O}(\mathbb{R}))$$

Theorem 8.20:

All sets in the Borel sigma algebra are in the Lebesgue sigma algebra, i.e.

$$\mathcal{B} \subset \mathcal{L}$$

✎ Proof.

□

Corollary 8.21:

The triple $(\mathbb{R}, \mathcal{B}, \mu)$ is a measure space, where μ is the Lebesgue measure.

However, there are Lebesgue measurable sets that are **not** Borel measurable.

Theorem 8.22:

Let C be the Cantor middle third set. There exists a subset $N \subset C$ such that $N \notin \mathcal{B}$.

Corollary 8.23:

$\mathcal{L} \not\subset \mathcal{B}$.

Corollary 8.24:

The measure space $(\mathbb{R}, \mathcal{B}, \mu)$ is not complete.

Chapter 9

Lebesgue Integration

Now that we have a notion of measure on $\mathbb{R}$ we can define Lebesgue integration. Our approach is the following:

1. Define the integrals of non-negative 'simple functions'.
2. Extend this definition to more general non-negative functions through a limiting process.
3. Extend this definition to more general (possibly negative) functions by decomposing them into a positive and negative part.

We adopt the convention that $0 \times \infty = \infty \times 0 = 0$.

9.1 Simple functions

Definition 9.1: Indicator functions

Let $E \subset \mathbb{R}$. The **indicator** function of E is the function $\mathbf{1}_E : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathbf{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

Recall in Example 6.2 we proved that the indicator function of the rational numbers was **not** Riemann integrable.

Definition 9.2: Simple functions

A function $f : \mathbb{R} \to \mathbb{R}$ is **simple** if for some $m \in \mathbb{N}$

$$f(x) = \sum_{i=1}^{m} \mathbf{1}_{E_i}(x)\, a_i \qquad \forall x \in \mathbb{R}$$

where $a_i \in \mathbb{R}$ and the sets E_i

- are Lebesgue measurable ($E_i \in \mathcal{L}$), and
- are pairwise disjoint, ($E_i \cap E_j = \emptyset$ for $i \neq j$).

Definition 9.3: Lebesgue integral of simple functions

Let $f:\mathbb{R}\to\mathbb{R}$ be a **non-negative** simple function of the form

$$f(x)=\sum_{i=1}^{m}\mathbf{1}_{E_i}(x)\,a_i \qquad \forall x\in\mathbb{R}.$$

The **Lebesgue integral** of f is defined by

$$\int_{\mathbb{R}} f\,\mathrm{d}\mu := \sum_{i=1}^{m}\mu(E_i)\,a_i.$$

Note that this integral can be infinite.

✎ **Example 9.4.**
The Lebesgue integral of the indicator function $\mathbf{1}_{\mathbb{Q}}$ is

$$\int_{\mathbb{R}}\mathbf{1}_{\mathbb{Q}}\,\mathrm{d}\mu = 0$$

✎ **Example 9.5.**
The function

$$f(x)=\begin{cases}3 & 1\le x\le 4\\ 5 & 4<x\le 8\\ 0 & \text{otherwise}\end{cases}$$

has Lebesgue integral

$$\int_{\mathbb{R}} f\,\mathrm{d}\mu = 29.$$

Lemma 9.6:

If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are simple functions then $f + g$ is a simple function.

✎ Proof.

□

9.2 Measurable functions

The functions of interest for the Lebesgue integral are the following:

Definition 9.7: Measurable function

A function $f : \mathbb{R} \to \mathbb{R}$ is **measurable** if
$$f^{-1}((a, b)) \in \mathcal{L}$$
for all open intervals $(a, b) \subset \mathbb{R}$.

In fact, this implies that $f^{-1}(B) \in \mathcal{L}$ for all $B \in \mathcal{B}$.

Lemma 9.8:

If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are measurable functions then the following functions are measurable

- $f^+ = \max(f, 0)$
- $f^- = \max(-f, 0)$

✎ Proof.

□

Lemma 9.9:

- Simple functions are measurable.
- Continuous functions are measurable.

✎ Proof.

□

Lemma 9.10:

If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are measurable, then $f + g$ and fg are measurable.

✎ Proof.

□

9.2.1 Measurable functions taking infinite values

Now, we consider functions $f : \mathbb{R} \to [-\infty, \infty]$ which can explicitly take infinite values. For example

$$f(x) = \begin{cases} x^{-2} & x \neq 0 \\ \infty & x = 0. \end{cases}$$

Definition 9.11:

A function $f : \mathbb{R} \to [-\infty, \infty]$ is **measurable** if $f^{-1}((a, \infty]) \in \mathcal{L}$ for all $a \in \mathbb{R}$.

This definition is compatible with Definition 9.7:

Lemma 9.12:

A function $f : \mathbb{R} \to \mathbb{R}$ is measurable in the sense of Definition 9.7 if and only if it is measurable in the sense of Definition 9.11.

✎ Proof.

□

Lemma 9.13:

If $f_n : \mathbb{R} \to [-\infty, \infty]$ is a sequence of measurable functions then the functions

$$g = \sup_{n \geq 1} f_n \qquad h = \limsup_{n \to \infty} f_n$$

are measurable.

✎ Proof.

□

Non-negative measurable functions can be approximated by simple functions:

Theorem 9.14:

If $f : \mathbb{R} \to [0, \infty]$ is a measurable function then there exists a sequence of simple functions $s_k : \mathbb{R} \to \mathbb{R}$ such that

- $0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq f(x)$ for all $x \in \mathbb{R}$, and
- $\lim_{k\to\infty} s_k(x) = f(x)$ for all $x \in \mathbb{R}$.

✎ Proof.

□

9.2.2 Integrating measurable functions

Non-negative measurable functions

First we deal with non-negative measurable functions.

Definition 9.15:

Let $f\colon \mathbb{R} \to [0, \infty]$ be a measurable function. The Lebesgue integral of f is defined by

$$\int_{\mathbb{R}} f \, \mathrm{d}\mu := \sup \int_{\mathbb{R}} s \, \mathrm{d}\mu \tag{9.1}$$

where the supremum is taken over all simple functions s such that $0 \leq s(x) \leq f(x)$ for all $x \in \mathbb{R}$.

Note that the supremum in (9.1) may be infinite.

We can define integration over a subset of $\mathbb{R}$.

Definition 9.16:

Let $f\colon \mathbb{R} \to [0, \infty]$ be a measurable function and $E \subset \mathbb{R}$ be a Lebesgue measurable set $E \in \mathcal{L}$. We define

$$\int_E f \, \mathrm{d}\mu = \int_{\mathbb{R}} \mathbf{1}_E f \, \mathrm{d}\mu.$$

Theorem 9.17: Properties of integration

Let $f, g\colon \mathbb{R} \to [0, \infty]$ be measurable functions, let $A, B \in \mathcal{L}$, and let $c \geq 0$ be a constant.

1. if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$ then $\int_{\mathbb{R}} f \, \mathrm{d}\mu \leq \int_{\mathbb{R}} g \, \mathrm{d}\mu$
2. if $A \subset B$ then $\int_A f \, \mathrm{d}\mu \leq \int_B f \, \mathrm{d}\mu$
3. $\int_{\mathbb{R}} cf \, \mathrm{d}\mu = c \int_{\mathbb{R}} f \, \mathrm{d}\mu$
4. if $f(x) = 0$ for all $x \in A$ then $\int_A f \, \mathrm{d}\mu = 0$
5. if $\mu(A) = 0$ then $\int_A f \, \mathrm{d}\mu = 0$.

Proof.

□

Measurable functions

If $f:\mathbb{R}\to[-\infty,\infty]$ is a measurable function then

- $f^+ := \max(f,0)$ and $f^- := \max(-f,0)$ are both non-negative, measurable functions,
- $|f| = f^+ + f^-$ is a non-negative, measurable function, and
- $f = f^+ - f^-$

Definition 9.18:

Let $f:\mathbb{R}\to[-\infty,\infty]$ be a measurable function. If the integral

$$\int_{\mathbb{R}} |f| \, d\mu < \infty$$

then we say that f is **Lebesgue integrable**, write $f \in L^1(\mathbb{R})$, and define the Lebesgue integral of f to be

$$\int_{\mathbb{R}} f \, d\mu := \int_{\mathbb{R}} f^+ \, d\mu - \int_{\mathbb{R}} f^- \, d\mu.$$

9.3 Lebesgue Integration Theorems

It will be useful to introduce the following family of measures:

Lemma 9.19:

Let $s:\mathbb{R}\to[0,\infty]$ be a simple function. The map $\nu:\mathcal{L}\to\mathbb{R}$ defined by

$$\nu(E) = \int_E s \, d\mu$$

is a measure.

✎ Proof.

□

Theorem 9.20: The Monotone Convergence Theorem

Let $f : \mathbb{R} \to [0, \infty]$ be a function and $f_n : \mathbb{R} \to [0, \infty]$ be a sequence of measurable functions such that

- $0 \leq f_1(x) \leq f_2(x) \leq \cdots$ for every $x \in \mathbb{R}$,
- $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$

then f is measurable and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \int_{\mathbb{R}} f \, d\mu$$

✎ Proof.

□

Corollary 9.21:

If $f : \mathbb{R} \to [0, \infty]$ is a measurable function then there exists a sequence of simple functions $s_k : \mathbb{R} \to \mathbb{R}$ such that

- $0 \le s_1(x) \le s_2(x) \le \cdots \le f(x)$ for all $x \in \mathbb{R}$
- $\lim_{k\to\infty} s_k(x) = f(x)$ for all $x \in \mathbb{R}$, and

$$\lim_{k\to\infty} \int_{\mathbb{R}} s_k(x)\ \mathrm{d}\mu = \int_{\mathbb{R}} f \mathrm{d}\mu$$

Theorem 9.22: Lebesgue integration is linear

Let $f, g \in L^1(\mathbb{R})$ and $a, b \in \mathbb{R}$ then $f + g \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} af + bg\ \mathrm{d}\mu = a \int_{\mathbb{R}} f\ \mathrm{d}\mu + b \int_{\mathbb{R}} g\ \mathrm{d}\mu$$

✎ Proof.

□

Theorem 9.23: The Dominated Convergence Theorem

Suppose $f_n : \mathbb{R} \to [-\infty, \infty]$ is a sequence of measurable functions such that the limit

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists for all $x \in \mathbb{R}$.
If there is a function $g \in L^1(\mathbb{R})$ such that

$$|f_n(x)| \leq g(x) \qquad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$$

then

$$f \in L^1(\mathbb{R}),$$
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f| \, d\mu = 0,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \int_{\mathbb{R}} f \, d\mu.$$

9.4 Relationship with Riemann integration

Theorem 9.24:

If $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable then $\mathbf{1}_{[a,b]} f$ is Lebesgue integrable and

$$\mathcal{R}\int_a^b f(x)\, \mathrm{d}x = \int_{\mathbb{R}} \mathbf{1}_{[a,b]} f \, \mathrm{d}\mu.$$

✎ Proof.

□