MSO3120
Complex Analysis

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Contents

Preparation and Other Reading

> "I am always doing that which I can not do, in order that I may learn how to do it."
>
> Pablo Picasso

These notes will introduce complex analysis. This subject is fundamental to many areas of mathematics and has a long history dating back to the mid 19th century. Many important mathematicians including Cauchy and Riemann initiated the work and you will see their names aligned to results throughout these notes.

There are many textbooks on complex analysis that might help you to understand concepts and delve deeper into areas not covered, the main recommendations are:

- Garling, D. J. H., "A Course in Mathematical Analysis, volume III", [3].

  This is a new textbook; it covers all of the topics, and extends some of what we will go over. This is a more modern book but it does avoid some of the geometric insights of some of the concepts. This is largely because the author is not a native complex analyst like the books below. But at least he is still alive.

- Ahlfors, L. V., "Complex Analysis", [1].

  This is considered one of the classics in complex analysis. Lars Ahlfors was one of the original Fields medal winners and he revolutionised many areas of complex analysis by introducing a geometric view of results. He is considered by many (including me) to be one of the greatest complex analysts. This book covers all of the module and is written by a master.

- Rudin, W., "Real and Complex Analysis", [7].

  This is another classic, including two sections covering real and complex analysis separately. This book departs a little from what we learn in this module and includes areas of current research.

- Remmert, R, "Theory of Complex Functions", [6]. I'll not reference this book very much but there's a copy in the library. It covers everything we will do in this module and includes quite a lot of detail (and some history of the authors of much of the work).

Chapter 1 : Complex Numbers, Complex Differentiation and Power Series

"I have also learned not to take glory in the difficulty of a proof: difficulty means we have not understood. The idea is to be able to paint a landscape in which the proof is obvious."

Alexander Grothendieck, quoted by Pierre Deligne, *Notices of the AMS* 63 (2016), p. 250

1.1 Complex numbers

The field of complex numbers is represented by the symbol $\mathbb{C}$. Elements of $\mathbb{C}$ are normally written in the form

$$a + bi \qquad i^2 = -1$$

where $a, b \in \mathbb{R}$. If we treat i as a variable then we may define arithmetic in $\mathbb{C}$ in the same way as we define arithmetic of polynomials in $\mathbb{R}[i]$, with the additional assumption that $i^2 = -1$. In particular we have

$$(a + bi) + (a' + b'i) = (a + a') + (b + b')i, \qquad \text{and}$$
$$(a + bi)(a' + b'i) = (aa' - bb') + (ab' + a'b)i.$$

The value of a in the complex number $a + bi$ is called the real part and denoted $\mathrm{Re}(a + bi)$. The value of b is called the imaginary part and denoted $\mathrm{Im}(a + bi)$.

Given a complex number we can describe it using the exponential form

$$a + bi = re^{i\theta}$$

where r and θ are related to a and b in the following way

$$a = r\cos\theta \qquad b = r\sin\theta.$$

These follow from Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

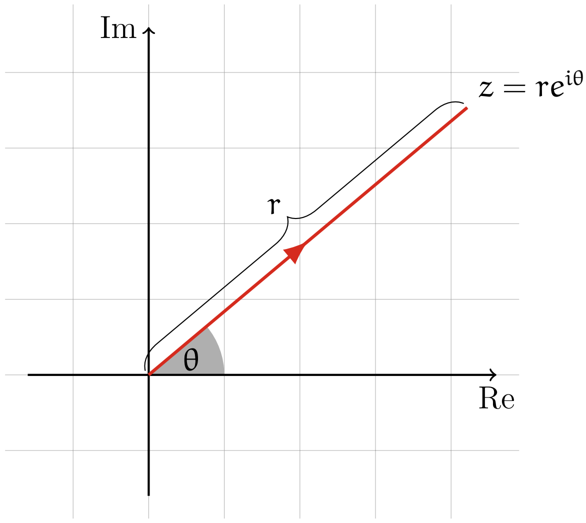


Figure 1.1: The exponential form of a complex number

The value r is often written as $|\cdot|$ and called the modulus or absolute value, for example $|1+i| = \sqrt{2}$. The value of θ is called the argument and denoted by $\arg$. It should be noted here that the argument is not uniquely defined since any multiple of 2π can be added or subtracted to produce the argument of the same complex number.

This representation is often called the exponential representation and is more convenient in many ways. For example we have the product formula:

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}.$$

For example it follows that

$$\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

We will normally reserve the letters z and w for complex numbers and x and y for real numbers.

Finally, if $z = a + ib$ then the complex conjugate is $\overline{z} = a - ib$. In exponential form

$$\overline{re^{i\theta}} = re^{-i\theta}.$$

Geometrically the conjugate of a complex number represents the reflection in the real axis of the Argand diagram.

We collect here properties of the conjugate and modulus of complex numbers.

Proposition 1.1. *Suppose $z, w \in \mathbb{C}$ then the following hold.*

1. $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$, $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$
2. $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$
3. $|z|^2 = z\bar{z}$
4. $|z + w| \leq |z| + |w|$ *(the Triangle inequality)*
5. $-|z| \leq \operatorname{Re} z \leq |z|$
6. $-|z| \leq \operatorname{Im} z \leq |z|$

1.2 The field of complex numbers

The complex number field first came about as the smallest splitting field over $\mathbb{R}$. It can be shown using the techniques you've learned in MSO3110 Advanced Algebra that

$$\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1).$$

This follows from the next theorem. We will prove this at the end of the course as a corollary to Liouville's theorem.

Theorem 1.2 – The fundamental theorem of algebra

Let $p(x) \in \mathbb{C}[x]$ have $\deg p \geq 1$. Then there is $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$ and p can be factorised as follows

$$p(x) = (x - \alpha)q(x)$$

where $q(x) \in \mathbb{C}[x]$.

The value α is called a root of the polynomial. Although this theorem only guarantees a single factor, it can be proved by induction that a polynomial $p(x)$ with $\deg p = n$ has exactly n roots (not necessarily distinct).

The solutions to equations of the form $z^n - a = 0$ are of particular importance in complex analysis. Finding solutions to this equation is equivalent to find the nth roots of the number a. In MSO1110 Vectors and Matrices you learned about the roots of unity, these are the solutions to the equation $z^n - 1$. It was shown that the n distinct roots are

$$\{1, \omega, \omega^2, \dots \omega^{n-1}\} \qquad \text{where } \omega = e^{2\pi i/n}.$$

These are drawn on the left in Figure 1.2.

In general it is possible to find the nth roots of any number using the same technique.

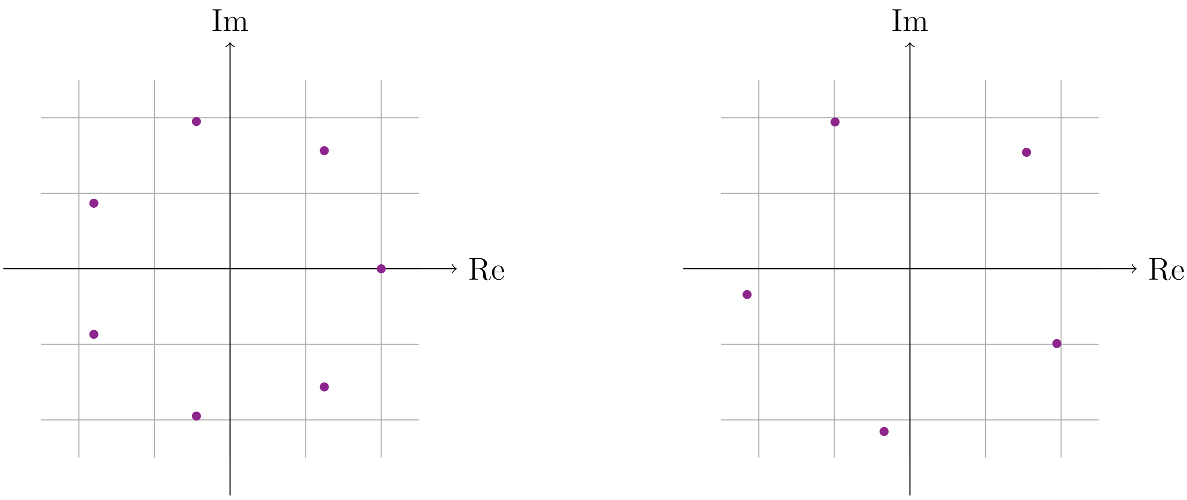


Figure 1.2: (Left) roots of unity with $n = 7$ and (right) roots of $(1 + i)$ with $n = 5$.

Example 1. Solve $z^n - (1 + i) = 0$.

First we write $1 + i = \sqrt{2}e^{\pi i/4}$. Letting $\xi = re^{i\theta}$ be a root, then we want to solve the equation

$$\xi^n = r^n e^{in\theta} = \sqrt{2}e^{\pi i/4}.$$

It follows that $r^n = \sqrt{2}$ so that $r = 2^{1/(2n)}$. To find values of θ we note that

$$e^{\pi i/4} = e^{\pi i/4 + 2\pi k i}$$

for any $k \in \mathbb{Z}$. Therefore if we let $k = 0, 1, 2, 3, \ldots, n-1$ and solve the equation $e^{in\theta} = e^{\pi i/4+2\pi k i}$ we find

$$\theta \in \left\{ \frac{\pi}{4n} + \frac{2\pi k}{n} : k = 0, 1, \ldots, n = 1 \right\}.$$

The solutions to $z^n - (1 + i) = 0$ are, then,

$$\left\{ 2^{1/(2n)} e^{\frac{\pi}{4n} + \frac{2\pi k}{n}} : k = 0, 1, \ldots, n - 1 \right\}.$$

In the right-hand diagram in Figure 1.2 these roots are plotted. They are in a familiar pattern of a regular polygon with n sides.

1.3 The geometry of complex numbers

Complex analysis relies very much on geometric intuition, indeed the domain of definition of a function – as you will shortly see – is given much higher status than it is in real analysis and functions

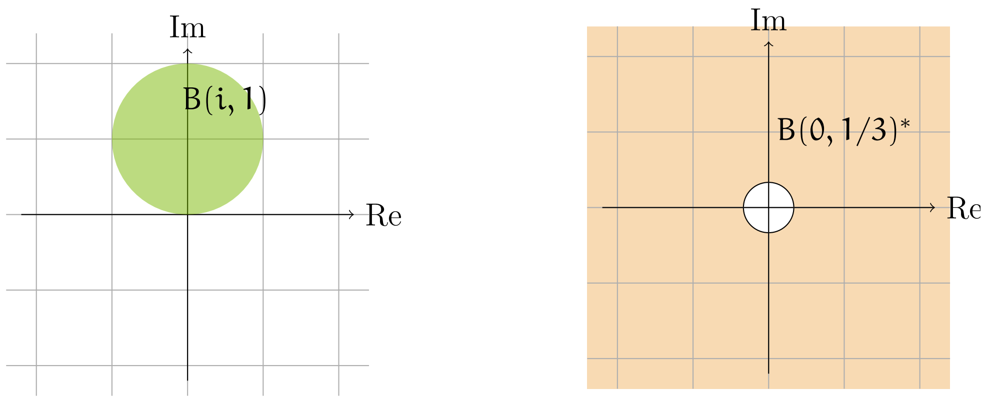


Figure 1.3: The sets in Examples 2 and 3

are studied much more in terms of their mapping properties. In this section we'll look at some important algebraic definitions of sets in $\mathbb{C}$.

Example 2. Sketch the set $B(i, 1) = \{z \in \mathbb{C} \colon |z - i| < 1\}$.

The quantity $|z - i|$ represents the distance from the variable z to the point i on the complex plane. This set consists of all the points closer than 1 to i and is the (open) disk drawn on the left in Figure 1.3.

In general a set of the form

$$B(a, r) = \{z \in \mathbb{C} \colon |z - a| < r\}$$

represents the open disk of radius r centred at the point a.

Related to the disk, the following are the *circle* and the *exterior of a disk*, respectively:

$$\partial B(a, r) = \{z \in \mathbb{C} \colon |z - a| = r\}, \quad \text{and} \qquad B(a, r)^* = \{z \in \mathbb{C} \colon |z - a| > r\}.$$

Example 3. Sketch the set $B(0, 1/3)^* = \{z \in \mathcal{C} \colon |z| > 1/3\}$

This time this set is the exterior of the disk of radius $1/3$ centred at the origin. A sketch of this is given on the right in Figure 1.3, it is the unbounded region external to the disk.

It is possible to combine these sets to form other examples. In Figure 1.4 a number of examples

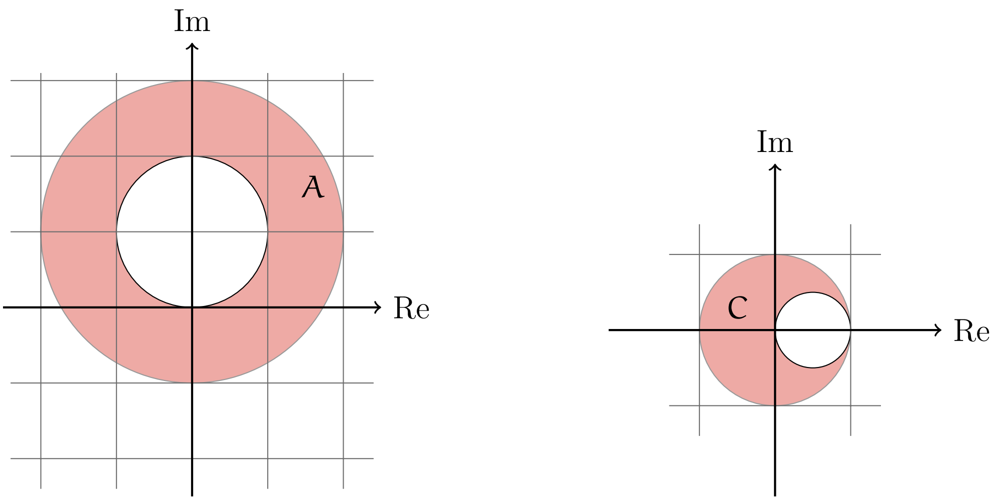


Figure 1.4: Other sets in $\mathbb{C}$: an annulus and a crescent

are given, they are respectively

$$A = \{z \in \mathbb{C} \colon 1 < |z - i| < 2\}$$
$$C = \{z \in \mathbb{C} \colon |z| < 1\} \setminus \{z \in \mathbb{C} \colon |z - 1/2| < 1/2\}$$

It will be useful to define straight lines in $\mathbb{C}$. The easiest way to do this is to simply note that a set such as

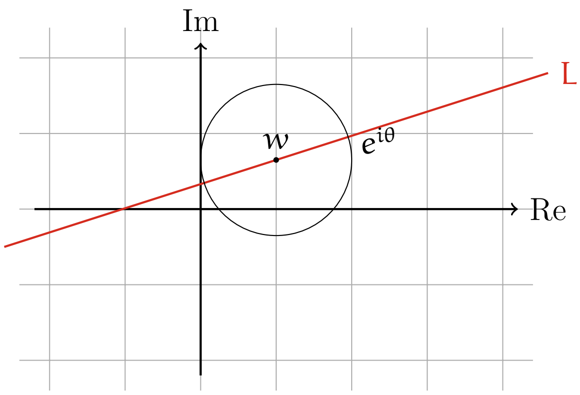
$$\{xe^{i\theta} \colon x \in \mathbb{R}\}$$

defines a straight line extending, in both directions, towards infinity. The line passes through the origin and is in the same direction as $e^{i\theta}$. If we want to define a line in general that passes through a point $w \in \mathbb{C}$ in the direction $e^{i\theta}$ we have

$$L = \{w + xe^{i\theta} \colon x \in \mathbb{R}\}.$$

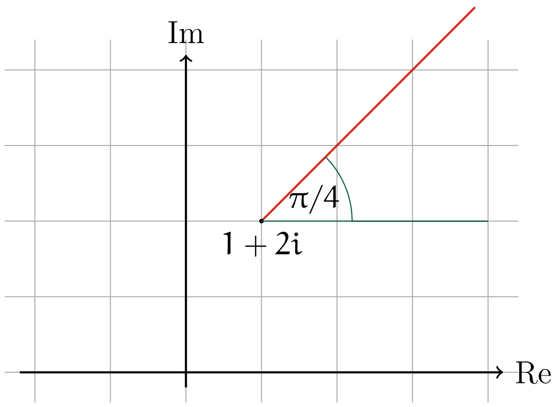
Note that the direction $e^{i\theta}$ can be replaced by $e^{-i\theta}$, i.e.

$$\{w + xe^{i\theta} \colon x \in \mathbb{R}\} = \{w + xe^{-i\theta} \colon x \in \mathbb{R}\}.$$



Following from the definition of a line above we can define a ray, that is a half-line pointing in a particular direction. For example the line emanating from the point $1 + 2i$ in the direction $\pi/4$ is drawn in the following diagram. We can write this as

$$\{z\colon \arg(z - (1 + 2i)) = \pi/4\}$$



In general a ray starting at a point a and heading in the direction θ can be described as

$$\{z\colon \arg(z - a) = \theta\}.$$

1.4 The metric space of complex numbers

The complex numbers $\mathbb{C}$ form a metric space with the metric

$$d(z, w) = |z - w|.$$

We define the mapping

$$\begin{array}{rcl} \mathcal{J} \;:\; \mathbb{C} & \to & \mathbb{R}^2 \\ a+ib & \mapsto & \begin{pmatrix} a \\ b \end{pmatrix}. \end{array} \tag{1.1}$$

The mapping $\mathcal{J}$ is a metric space isometry from the metric space $(\mathbb{C}, d)$ to the metric space $(\mathbb{R}^2, d_2)$ where d_2 is the standard Euclidean metric. The definitions of open sets, closed sets, continuous functions etc. on $(\mathbb{C}, d)$ can be thought of in terms of the metric space $(\mathbb{R}^2, d_2)$. In particular, by the Heine-Borel theorem, a set A in $\mathbb{C}$ is compact if and only if it is closed and bounded.

For example an ϵ-neighbourhood of a point $a \in \mathbb{C}$ is the set $B(a, \epsilon)$ and is an open disk centred at a with radius $\epsilon > 0$.

A sequence of complex numbers $z_0, z_1, z_2, \ldots$ will be denoted by $(z_n)_{n=0}^{\infty}$. A sequence is said to converge to the limit z if for each $\epsilon > 0$ there is a $N > 0$ such that

$$n > N \qquad \Rightarrow \qquad |z_n - z| < \epsilon.$$

Familiar rules of real sequences apply to complex sequences. For example limits are unique, the limit of the sum of two (or more) sequences is the sum of the limits. Furthermore a convergent sequence is bounded. Recall that a set is bounded in a metric space if it is contained in an open ball $B(a, r)$. In the complex metric space this is equivalent to the existence of a $r > 0$ such that each element of the set has absolute value less than r.

Proposition 1.3. *If (z_n) is a convergent complex-valued sequence then it is bounded.*

Proof. Let $f(z) = |z| = d(0, z)$ then f is a continuous function from $\mathbb{C}$ to $\mathbb{R}$ by standard results from metric spaces. It follows that if $z_n \to z$ as $n \to \infty$ then

$$|z_n| \to |z| \qquad \text{as } n \to \infty.$$

Since this is a real-valued convergent sequence it is bounded and hence (z_n) is bounded. $\square$

Other common results transfer from real analysis via the next result.

Proposition 1.4. *Suppose $z_n = x_n + iy_n$ for all $n \geq 0$. The sequence (z_n) converges if and only if the sequences (x_n) and (y_n) converge. When the limits exist*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n.$$

The sequences (x_n) and (y_n) can be described using the isometry (1.1) as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathcal{J}(z_n) \qquad \text{for all } n.$$

Proof. Suppose that (z_n) converges to $z = x + iy$. Then since $\operatorname{Re} w \leq |\operatorname{Re} w| \leq |w|$ for any $w \in \mathbb{C}$ we have that

$$\begin{aligned} d(x_n, x) &= |x_n - x| \\ &= |\operatorname{Re}(z_n - z)| \\ &\leq |z_n - z| \\ &= d(z_n, z) \end{aligned}$$

It follows that if $d(z_n, z) \to 0$ then $d(x_n, x) \to 0$ as $n \to \infty$. The same argument with the imaginary part instead shows that $d(y_n, y) \leq d(z_n, z)$ and so, again, if $d(z_n, z) \to 0$ then $d(y_n, y) \to 0$ as $n \to \infty$.

To prove the converse is easier. Suppose that

$$x_n \to x, \qquad \text{and} \qquad y_n \to y$$

as $n \to \infty$. Then, with $z = x + iy$,

$$\begin{aligned} d(z_n, z) &= |x_n + iy_n - (x + iy)| \\ &= |(x_n - x) + i(y_n - y)| \\ &\leq |(x_n - x)| + |i|\,|(y_n - y)| \\ &= d(x_n, x) + d(y_n, y). \end{aligned}$$

Since $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$ as $n \to \infty$ we have that $d(z_n, z) \to 0$ and the lemma is proved. □

Lastly we have the following result that will be used throughout.

Proposition 1.5. *A complex sequence (z_n) tends to L if and only if the real sequence $(|z_n - L|)$ tends to 0.*

Proof. This is simply a restatement of the definition. □

It is not possible to define the lim sup and lim inf of a complex sequence since $\mathbb{C}$ does not have an ordering by $\leq$ like the set of real numbers does and therefore there is no concept of a supremum or infimum for $\mathbb{C}$.

A series of complex numbers $\sum_{n=0}^{\infty} a_n$ is said to converge if the sequence of partial sums

$$s_N = \sum_{n=0}^{N} a_n$$

converge as a sequence. A series is said to converge absolutely if

$$\sum_{n=0}^{\infty} |a_n|$$

converges. Since the latter is a real-valued series the tests of convergence that you have come across elsewhere can be used to determine if a given series converges absolutely or not. Although we won't prove it in this module it can be shown quite easily that if a series converges absolutely then it converges.

1.5 Power series

Power series are particularly important in complex analysis, these are functions of the form

$$\sum_{n=0}^{\infty} a_n(z-a)^n$$

A power series converges for the value z whenever the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(z-a)^n$$

exists.

In MSO2120 Mathematical Analysis you learned that a power series converges absolutely for all $|z-a| < R$ where R is defined by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

The value R is called the *radius of convergence* of the power series. It is possible for the limit above to be 0 in which case we say that $R = \infty$ and the power series converges for all values of z. Similarly it is possible for the limit to be infinite in which case the power series does not converge absolutely for any value of z (except a trivially).

The set in which a power series converges is therefore a disk

$$B(a, R) = \{z : |z-a| < R\}$$

(hence the use of 'radius' to describe R), Figure 1.5.

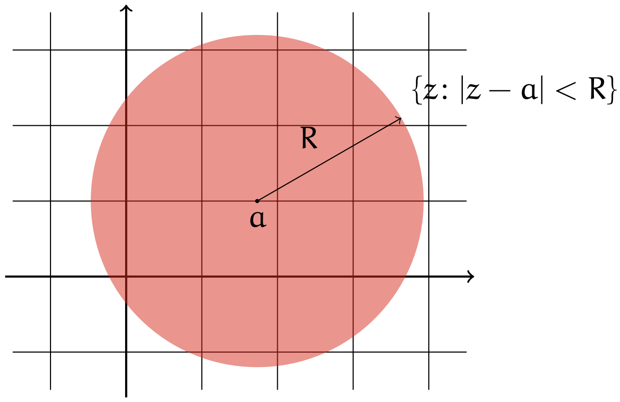


Figure 1.5: The disk centred at a with radius $R > 0$

1.6 *Differentiation*

The main object of study in complex analysis is a complex differentiable function. We will see that the implications of complex differentiability are more restrictive than those of real differentiability. It is worth discussing briefly the definitions of both before we study them in more detail.

Suppose that $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))^T$ is a function from some open set $A \subset \mathbb{R}^2$ into $\mathbb{R}^2$. Here

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

If $\mathbf{f}$ is differentiable at each point in A then the derivative of $\mathbf{f}$ at $\mathbf{x}$ is given by

$$\nabla\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(\mathbf{x}) & \frac{\partial f_1}{\partial y}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x}(\mathbf{x}) & \frac{\partial f_2}{\partial y}(\mathbf{x}) \end{pmatrix}$$

Recall that this means that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - \nabla\mathbf{f}(\mathbf{x}_0)\mathbf{h}|}{|\mathbf{h}|} = 0$$

In other words the expression

$$\mathbf{f}(\mathbf{x}_0) + \nabla\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

is the best linear approximation to $\mathbf{f}(\mathbf{x})$ in a neighbourhood of $\mathbf{x}$.

On the other hand suppose instead that we have a function

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto f(z). \end{aligned}$$

We define the complex derivative of f at the point z_0 as

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

whenever this limit exists.

These two definitions of differentiation are not the same.

Example 4. Let $f(z) = |z|$. Find the real derivative of z at 1 and the complex derivative.

Given the isometry $\mathcal{J}$ defined in (1.1) we can always define the equivalent function $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f} = \mathcal{J} \circ f \circ \mathcal{J}^{-1}.$$

In this example we have that

$$\mathbf{f}(\mathbf{x}) = (|\mathbf{x}|, 0)^T.$$

We need to calculate this derivative at $\mathbf{x} = (1,0)^T = \mathcal{J}(1)$.

Now the partial derivatives of this function exist and are continuous at $(1,0)^T$ and so by Theorem 3.17 of Nick's notes we can write

$$\nabla\mathbf{f} = \begin{pmatrix} x(x^2+y^2)^{-1/2} & y(x^2+y^2)^{-1/2} \\ 0 & 0 \end{pmatrix}.$$

Evaluated at $(1,0)^T$ we have that

$$\nabla\mathbf{f}((1,0)^T) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand to calculate the complex derivative we need to find the limit as $h \to 0$ of the fraction

$$\frac{f(1+h) - f(1)}{h} = \frac{|1+h| - 1}{h}.$$

This limit does not exist. To see this suppose first that $h \to 0$ along the real line, then $|1+h| = 1+h$ and

$$\frac{|1+h|-1}{h} = \frac{1+h-1}{h} = 1.$$

On the other hand if $h \to 0$ along the imaginary axis, then for we can write $h = ih'$. Now

$$\frac{|1+h|-1}{z-1} = \frac{(1+(h')^2)^{1/2} - 1}{ih'}.$$

The limit as $h' \to 0$ of the fraction on the right is 0 which is different to the limit we found when z converged to 1 along the real line. Therefore the complex derivative doesn't exist.

This example begins to highlight the difficulty with defining complex differentiation. Namely the limit $h \to 0$ can approach from any direction. You encountered this type of issue when you looked at directional derivatives with Nick.

It is for this reason that complex analysis places a higher emphasis on the *domain* of a function than real analysis does. A set A in $\mathbb{C}$ is said to be *connected* if there is a path from any point to any other point in A.

Definition 1.6. A set Ω in $\mathbb{C}$ is a *domain* if it is open and connected.

Throughout these complex analysis notes, unless otherwise stated, the set Ω will be a domain. It is usual to define complex-valued functions from a domain to the complex plane. For example

$$\begin{aligned} f \; : \; \Omega &\to \mathbb{C} \\ z &\mapsto f(z). \end{aligned}$$

Things to know

- Complex numbers – addition, multiplication and division;
- Polar and exponential forms, and how to convert a complex number into any given form;
- Complex conjugates;
- The statement of Proposition 1.1;
- How to calculate the nth roots of a given complex number;
- Geometric descriptions of sets such as $\{z \in \mathbb{C} \colon |z - a| = r\}$;
- Definitions of convergence of complex sequences and series, how to find the limit of common sequences;
- The statements of Propositions 1.4 and 1.5.

Problems

1. The n roots of a complex number, $w = re^{i\theta}$ are
$$\omega_k = r^{1/n} e^{i(2\pi k+\theta)/n}, \qquad k = 0, 1, \ldots, n-1.$$
   (a) Find the 5th roots of -1
   (b) Find the cube roots of i
   (c) Find the cube roots of $-i$
   (d) Find the 4th roots of $1 + i$
   (e) Find the cube roots of $2 + 2\sqrt{3}i$
2. Sketch the following regions in $\mathbb{C}$
   (a) $\{z \in \mathbb{C} \colon |z + 1| < 2\}$
   (b) $\{z \in \mathbb{C} \colon \operatorname{Re}(z - 1/2) > 0\}$
   (c) $\{z \in \mathbb{C} \colon \operatorname{Re}(z - i) > 0\}$
   (d) $\{z \in \mathbb{C} \colon |z - i| > |z + i|\}$
   (e) $\{z \in \mathbb{C} \colon |z - i| > 2|z + i|\}$
3. Find the solution of $z^2 + 3iz + i = 0$. Why are the root *not* complex conjugates?
4. Let C be the set of real-valued matrices of the form
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$
   (a) Show that C is a vector space over $\mathbb{R}$ (show that it's closed under addition, and scalar multiplication)
   (b) Show that the matrices
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
   form a basis for C and hence that C is 2-dimensional.
   (c) Show that the map $\mathcal{H} \colon C \to \mathbb{C}$ defined by
$$\mathcal{H}(aI + bJ) = a + bi$$
   is bijective.

Chapter 2 : Complex Differentiation

> "Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."
>
> Bertrand Russell

2.1 *Holomorphic functions*

Complex analysis can be summarised as a study of complex-differentiable functions (or, more accurately, holomorphic functions). In the last part you saw that complex-differentiation is different to real-differentiation. We will start this topic by comparing complex and real-differentiation via the Cauchy-Riemann equations but end up highlighting the differences between the two.

First we need to define what we will be studying.

Definition 2.1 – Holomorphic functions

A complex-valued function f defined on a domain Ω is *holomorphic* if the derivative

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists for all values $z_0 \in \Omega$. We denote the set of holomorphic functions on Ω as $\mathcal{H}(\Omega)$. A function $f \in \mathcal{H}(\mathbb{C})$ is called *entire*.

It is important that whenever we discuss complex functions we state or assume that the function is defined on a domain (in the sense of Definition 1.6). In the words of Lars V. Ahlfors, [1]:

> "It is important to stress that the open set Ω is part of the definition. As a rule one should avoid speaking of a [holomorphic] function $f(z)$ without referring to a specific open set Ω on which it is defined."

Example 5. If p is a polynomial then $p \in \mathcal{H}(\mathbb{C})$. So p is entire.

Example 6. The function

$$f(z) = \frac{1}{z}$$

is in $\mathcal{H}(\Omega)$ where

$$\Omega = \{z\colon 1 < |z| < 2\}.$$

Here the derivative is

$$f'(z) = -\frac{1}{z^2}.$$

Example 7. The function

$$f(z) = \frac{1}{z-1}$$

is holomorphic on any domain that does not contain the point 1.

For example we could take the unit disc centred at the origin:

$$\Omega = B(0,1) = \{z\colon |z| < 1\}.$$

The derivative for this function is

$$f'(z) = -\frac{1}{(z-1)^2}.$$

For the remainder of these notes we will often refer to complex-differentiability simply as differentiability unless it is not clear whether I am talking about real or complex-differentiation.

Theorem 2.2

Suppose that f is defined in a neighbourhood of a point z_0, $B(z_0, \epsilon)$. Then f is differentiable at z_0 if and only if there is a complex valued function E defined on $B(0,\epsilon)\backslash\{0\}$ with $E(h)/h \to 0$ as $h \to 0$ and a $d \in \mathbb{C}$ such that

$$f(z_0 + h) = f(z_0) + dh + E(h).$$

In this case we have that $f'(z_0) = d$.

This theorem is very useful as it provides a method for determining when a given function is differentiable, and hence holomorphic. It will be used throughout these notes for exactly this purpose.

Proof. Suppose that f is differentiable at z_0 then let

$$e(h) = \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0). \tag{2.1}$$

This is defined for all $h \in B(0,\epsilon)\backslash\{0\}$. Furthermore $\lim_{h\to 0} e(h) = 0$ since f is differentiable at z_0. Letting

$$E(h) = he(h)$$

and rearranging (2.1) we get

$$f(z_0 + h) = f(z_0) + f'(z_0)h + E(h)$$

and $E(h)/h = e(h) \to 0$ as $h \to 0$.

Suppose, conversely, that

$$f(z_0 + h) = f(z_0) + dh + E(h)$$

where $E(h)$ is defined in a neighbourhood $B(0, \epsilon)\backslash\{0\}$ and $E(h)/h \to 0$ as $h \to 0$. Then again we can rearrange this so that

$$\frac{f(z_0 + h) - f(z_0)}{h} = d + \frac{E(h)}{h}.$$

It follows that

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h\to 0} \left(d + \frac{E(h)}{h} \right) \\ &= d. \end{aligned}$$

□

In general the standard rules for differentiation in calculus have equivalent counterparts in the complex plane, for example if $f(z) = z^n$ for some $n \in \mathbb{C}$ then $f'(z) = nz^{n-1}$. The product rule, chain rule and quotient rule hold when we replace real-differentiation with complex-differentiation. The examples above illustrate the importance of the domain as a crucial element of the definition of a holomorphic function and the same is true when we use the rules of differentiation. We need to make sure we are aware that the rules are only true as long as our functions are holomorphic.

Theorem 2.3 – Product Rule

Suppose $f \in \mathcal{H}(\Omega_1)$ and $g \in \mathcal{H}(\Omega_2)$ then for all $z_0 \in \Omega_1 \cap \Omega_2$

$$\frac{d}{dz}(fg)(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

Proof. Since both $f'(z_0)$ and $g'(z_0)$ exist we have, by Theorem 2.2, functions $E(h)$ and $F(h)$ such that

$$\begin{aligned} f(z_0 + h) &= f(z_0) + f'(z_0)h + E(h) \\ g(z_0 + h) &= g(z_0) + g'(z_0)h + F(h), \end{aligned}$$

where $\lim_{h\to 0} E(h)/h = \lim_{h\to 0} F(h)/h = 0$.

Now

$$\begin{aligned} f(z_0+h)g(z_0+h) &= \left(f(z_0)+f'(z_0)h+E(h)\right)\left(g(z_0)+g'(z_0)h+F(h)\right) \\ &= f(z_0)g(z_0)+f(z_0)g'(z_0)h+f(z_0)F(h) \\ &\qquad + f'(z_0)g(z_0)h+f'(z_0)g'(z_0)h^2+f'(z_0)F(h)h \\ &\qquad\quad + E(h)g(z_0)+E(h)g'(z_0)h+E(h)F(h) \\ &= f(z_0)g(z_0)+\left(f(z_0)g'(z_0)+f'(z_0)g(z_0)\right)h \\ &\qquad + f(z_0)F(h)+f'(z_0)g'(z_0)h^2 \\ &\qquad\quad + f'(z_0)F(h)h+E(h)g(z_0)+E(h)g'(z_0)h+E(h)F(h) \\ &= f(z_0)g(z_0)+\left(f(z_0)g'(z_0)+f'(z_0)g(z_0)\right)h+G(h) \end{aligned}$$

where

$$G(h) = f(z_0)F(h)+E(h)g(z_0)+h(f'(z_0)F(h)+E(h)g'(z_0))+f'(z_0)g'(z_0)h^2+E(h)F(h).$$

But

$$\frac{G(h)}{h} = f(z_0)\frac{F(h)}{h}+g(z_0)\frac{E(h)}{h}+(f'(z_0)F(h)+E(h)g'(z_0))+f'(z_0)g'(z_0)h+\frac{E(h)}{h}F(h).$$

Since each term converges to 0 in this expression, we have that $\lim_{h\to 0} G(h)/h = 0$ and by Theorem 2.2 the result is proved. □

We will also state here without proving it the chain rule.

Theorem 2.4 – Chain Rule

Suppose that $f\colon \Omega_1 \to \Omega_2$ is holomorphic, as is $g\colon \Omega_3 \to \Omega_1$. Then $f \circ g \in \mathcal{H}(\Omega_3)$ and

$$\frac{d}{dz} f \circ g(z_0) = g'(z_0)\, f'(g(z_0)), \qquad z_0 \in \Omega_3.$$

Example 8. The derivative of z^n is nz^{n-1} whenever $n \in \mathbb{N}$.

Let $h(z) = z^n$. To prove this note that when $n = 1$, $h(z) = z$ and so it follows immediately from Theorem 2.2 that $h'(z) = 1$.

Suppose that the result is true for k and let $h(z) = z^{k+1}$ then we write

$$h(z) = z^{k+1} = f(z)g(z)$$

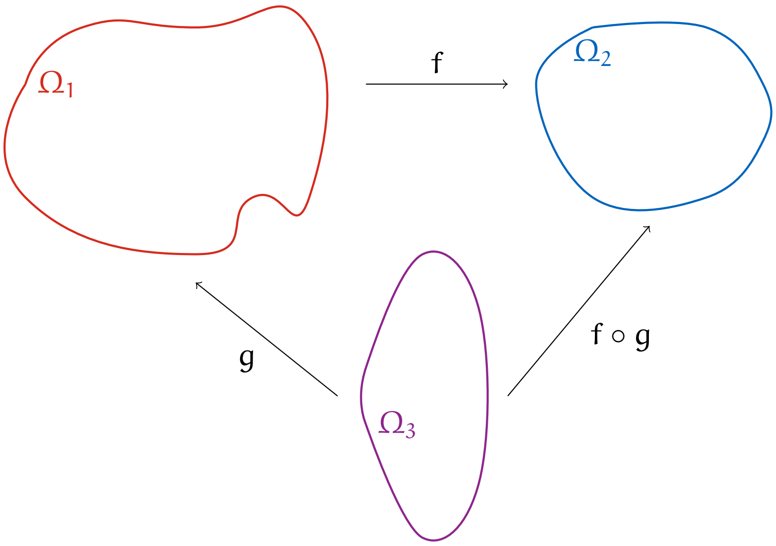


Figure 2.1: The Chain Rule

where $f(z) = z^k$ and $g(z) = z$. Then by the inductive step $f'(z) = kz^{k-1}$ and we have that $g'(z) = 1$. So by the Product Rule

$$f'(z) = kz^{k-1} \times z + z^k \times 1 = (k+1)z^k$$

as required.

2.2 The Cauchy-Riemann equations

Let $f\colon \mathbb{C} \to \mathbb{C}$ and define $\mathbf{f} = \mathcal{J} \circ f \circ \mathcal{J}^{-1}$ where $\mathcal{J}$ was defined in (1.1).

We saw in the last section that f is complex-differentiable at $z_0 = x_0 + iy_0$ if

$$f(z_0 + h) \approx f(z_0) + dh \tag{2.2}$$

for some $d \in \mathbb{C}$ and h close enough to 0.

Similarly $\mathbf{f}$ is real-differentiable at $\mathbf{x}_0 = (x_0, y_0)^T$ if

$$\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) \approx \mathbf{f}(x_0) + D\mathbf{h} \tag{2.3}$$

for some 2×2 square matrix D, and $\mathbf{h}$ close enough to $\mathbf{0}$.

If we write $d = a + ib$ in (2.2) then define

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

A calculation shows that

$$\mathcal{J}(dz) = D\mathcal{J}(z).$$

So if f satisfies (2.2) with derivative $d \in \mathbb{C}$ then $\mathbf{f}$ satisfies (2.3) with derivative D. That is, if f is complex-differentiable then $\mathbf{f}$ is real-differentiable. The converse is not, however, true: if $\mathbf{f}$ is real-differentiable it does not mean that f is complex-differentiable. In this section we will prove the Cauchy-Riemann equations that demonstrate that this relationship holds throughout the domain of definition of a holomorphic function.

Theorem 2.5 – The Cauchy-Riemann equations

Suppose that $f\colon \Omega \to \mathbb{C}$ is written as

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then

1. If f is holomorphic then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.4}$$

   at each point in Ω.

2. Conversely if u and v are continuously differentiable functions in Ω that satisfy (2.4) then f is holomorphic.

Proof. Since Ω is open we only need to prove the theorem in a neighbourhood of a given point $z_0 = x_0 + iy_0$.

We prove part 1 first. Since $f'(z_0)$ exists we have that

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

where the limit is taken in any direction. In particular if we restrict h to the real numbers, $h \in \mathbb{R}$, then $z_0 + h = (x_0 + h) + iy_0$ and

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{(u(x_0+h, y_0) + iv(x_0+h, y_0)) - (u(x_0, y_0) + iv(x_0, y_0))}{h} \\ &= \lim_{h\to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i \lim_{h\to 0} \frac{v(x_0+h, y_0) - v(x_0, y_0)}{h} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

Similarly we can take the limit along the imaginary axis so that $h = ik$ for $k \in \mathbb{R}$. Then $z_0 + h = x_0 + i(y_0 + k)$ and

$$\begin{aligned} f'(z_0) &= \lim_{k\to 0} \frac{(u(x_0, y_0+k) + iv(x_0, y_0+k)) - (u(x_0, y_0) + iv(x_0, y_0))}{ik} \\ &= \lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0, y_0)}{ik} + i \lim_{k\to 0} \frac{v(x_0, y_0+k) - v(x_0, y_0)}{ik} \\ &= -i \lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0, y_0)}{k} + \lim_{k\to 0} \frac{v(x_0, y_0+k) - v(x_0, y_0)}{k} \end{aligned}$$

Since $1/i = -i$,

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

These two limits are the same since $f'(z_0)$ exists and hence we have that

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

The Cauchy-Riemann equations (2.4) follow by equating real and imaginary parts.

To prove the converse recall that since u and v are continuously differentiable throughout Ω,

$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix}, \qquad \nabla v = \begin{pmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Moreover

$$\lim_{\mathbf{h}\to \mathbf{0}} \frac{|u(\mathbf{x}_0 + \mathbf{h}) - u(\mathbf{x}_0) - \nabla u\, \mathbf{h}|}{|\mathbf{h}|} = 0, \tag{2.5}$$

and

$$\lim_{\mathbf{h}\to \mathbf{0}} \frac{|v(\mathbf{x}_0 + \mathbf{h}) - v(\mathbf{x}_0) - \nabla v\, \mathbf{h}|}{|\mathbf{h}|} = 0, \tag{2.6}$$

where $\mathbf{h}$ and $\mathbf{x}_0$ are vector quantities in $\mathbb{R}^2$.

Let $h \in \mathbb{C}$ and define

$$E(h) = f(z_0 + h) - f(z_0) - dh$$

where

$$d = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

By Theorem 2.2 the result will follow if we show that $E(h)/h \to 0$ as $h \to 0$. Let us write $h = h_1 + ih_2$. We first take the real part of $E(h)$,

$$\begin{aligned}
\operatorname{Re} E(h) &= \operatorname{Re}(f(z_0 + h) - f(z_0) - dh) \\
&= \operatorname{Re}\bigg(u(x_0 + h_1, y_0 + h_2) + iv(x_0 + h_1, y_0 + h_2) - (u(x_0, y_0) + iv(x_0, y_0)) \\
&\qquad - \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h_1 + ih_2)\bigg) \\
&= u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) - \operatorname{Re}\left(\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h_1 + ih_2)\right) \\
&= u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) - \left(\frac{\partial u}{\partial x}h_1 - \frac{\partial v}{\partial x}h_2\right) \\
&= u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) - \frac{\partial u}{\partial x}h_1 + \frac{\partial v}{\partial x}h_2 \\
&= u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) - \frac{\partial u}{\partial x}h_1 - \frac{\partial u}{\partial y}h_2
\end{aligned}$$

by (2.4).

Now if we let $\mathbf{x}_0 = \mathcal{J}(z_0) = (x_0, y_0)^T$ and $\mathbf{h} = \mathcal{J}(h) = (h_1, h_2)^T$ then we may rewrite this as

$$\operatorname{Re} E(h) = u(\mathbf{x}_0 + \mathbf{h}) - u(\mathbf{x}_0) - \nabla u\, \mathbf{h}$$

then

$$\lim_{h \to 0}\left|\frac{\operatorname{Re} E(h)}{h}\right| = \lim_{|h| \to 0}\frac{|u(\mathbf{x}_0 + \mathbf{h}) - u(\mathbf{x}_0) - \nabla u\, \mathbf{h}|}{|\mathbf{h}|} = 0$$

by (2.5).

If instead we calculate the imaginary part of $E(h)$ above then we see that

$$\operatorname{Im} E(h) = v(\mathbf{x}_0 + \mathbf{h}) - v(\mathbf{x}_0) - \nabla v\, \mathbf{h}$$

and it follows by (2.6) that

$$\lim_{h \to 0}\left|\frac{\operatorname{Im} E(h)}{h}\right| = \lim_{|h| \to 0}\frac{v(\mathbf{x}_0 + \mathbf{h}) - v(\mathbf{x}_0) - \nabla v\, \mathbf{h}}{|\mathbf{h}|} = 0.$$

Since $\operatorname{Re} E(h)/h \to 0$ and $\operatorname{Im} E(h)/h \to 0$ as $h \to 0$, it follows by Proposition 1.4 that $E(h)/h \to 0$ as $h \to 0$. So f is differentiable at z_0 by Theorem 2.2. Since Ω is open this is true at all points. It follows that the function f is holomorphic. □

A number of consequences of the Cauchy-Riemann equations will demonstrate why they are important, the first being the following.

Corollary 2.6. *Suppose that* $f(x+iy) = u(x,y) + iv(x,y)$ *is holomorphic in a neighbourhood then*

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \end{aligned}$$

The first description of the derivative in this corollary gives us an easy way to find the derivative of a function since we can think of it as

$$f'(z) = \frac{\partial}{\partial x} f(z)$$

where $z = x + iy$. Similarly the second can be written as

$$f'(z) = -i\frac{\partial}{\partial y} f(z).$$

Example 9. Find the derivative of z^2.

Here we write this as

$$(x+iy)^2 = x^2 + 2ixy - y^2$$

then

$$\begin{aligned} \frac{d}{dz} z^2 &= \frac{\partial}{\partial x}(x^2 + 2ixy - y^2) \\ &= 2x + 2iy = 2(x+iy) \\ &= 2z. \end{aligned}$$

Let f be a holomorphic function and define

$$\mathbf{f} = \mathcal{J} \circ f \circ \mathcal{J}^{-1}$$

be the equivalent real-valued function $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$. With the notation of Theorem 2.5 we may write this function as

$$\begin{aligned} \mathbf{f} \ : \quad \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x,y)^T &\mapsto \big(u(x,y), v(x,y)\big)^T. \end{aligned}$$

The derivative is

$$\nabla \mathbf{f} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

However, note that the Cauchy-Riemann equations state that we may write this as

$$\nabla \mathbf{f} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}. \tag{2.7}$$

We therefore have

$$\begin{aligned} \det \nabla \mathbf{f} &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= |f'(z)|^2. \end{aligned}$$

We have proved the following.

Corollary 2.7. *Let* f *and* $\boldsymbol{f}$ *be defined as above and* $z = x + iy$*. If* f *is holomorphic then*

$$\det \nabla \boldsymbol{f}(x, y) = |f'(z)|^2.$$

Before moving on we will, on occasion, need the Cauchy-Riemann equations in polar form. To find these require some elementary work using the equations

$$x = r\cos\theta \qquad y = r\sin\theta.$$

The chain rule for partial derivatives then gives the following

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -\frac{1}{r}\frac{\partial v}{\partial r}. \tag{2.8}$$

2.3 Conformal mappings

An important subclass of holomorphic functions that we'll discuss here are conformal mappings.

Definition 2.8 – Conformal mappings

A function $f \in \mathcal{H}(\Omega)$ is called a *conformal mapping* if for all $z \in \Omega$, $f'(z) \neq 0$.

As a consequence of the last corollary and the Inverse Function Theorem we have the following result.

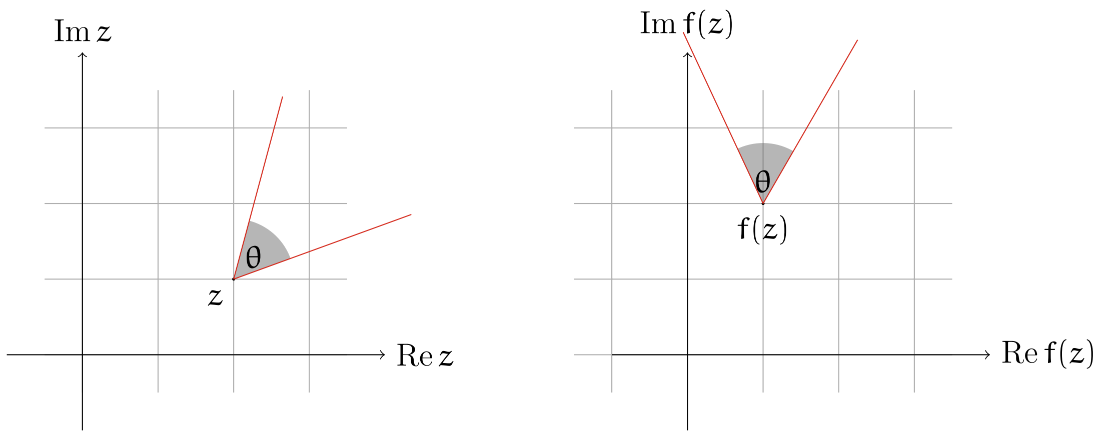


Figure 2.2: Conformal mappings

Corollary 2.9. *Suppose that* f *is a conformal mapping of a domain* Ω *then in a neighbourhood of each* $z \in \Omega$*,* f *is bijective and its inverse is also holomorphic.*

The corollary claims only that a conformal mapping is bijective in a neighbourhood of each point. It is a far-reaching fact of complex analysis that this cannot be extended to the entire domain Ω, i.e. we cannot conclude that f is a bijective mapping for all of Ω, unless Ω is simply connected. When Ω is simply connected a holomorphic function is conformal if and only if it is a bijection.

Recall from MSO2120 or from MSO2110 the group $\mathbf{O}_2(\mathbb{R})$ consists of isometries of $\mathbb{R}^2$, these are 2×2 square matrices that satisfy $AA^T = A^TA = I$. Note that this implies that $\det A = \pm 1$. Isometries with $\det A = 1$ are called *orientation preserving* and are denoted by $\mathbf{SO}_2(\mathbb{R})$, the special orthogonal group. Matrices in $\mathbf{SO}_2(\mathbb{R})$ have the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in [0, 2\pi).$$

They represent a rotation anticlockwise around the origin by an angle of θ.

In (2.7) we showed that a consequence of the Cauchy-Riemann equations is that if $f \in \mathcal{H}(\Omega)$ then at each point of Ω, $\nabla \mathbf{f}$ is of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad a, b \in \mathbb{R}.$$

This means that at each point in Ω

$$\nabla \mathbf{f} = \rho A \qquad \text{where } \rho \geq 0, \quad A \in \mathbf{SO}_2(\mathbb{R}).$$

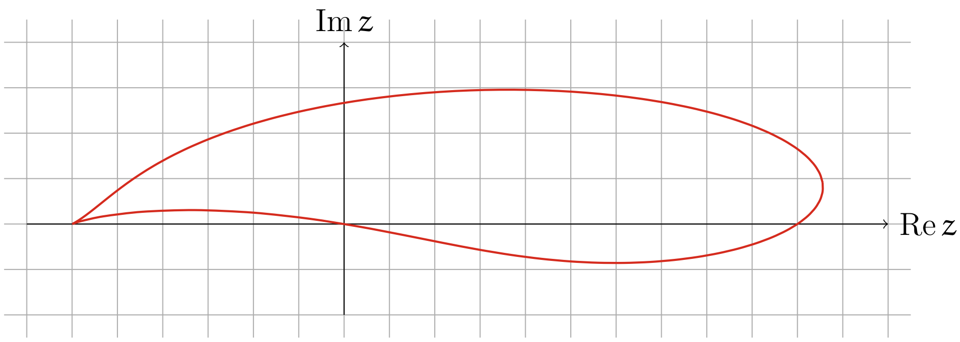


Figure 2.3: The Joukowsky aerofoil

If f is, in addition, a conformal mapping then we must have $\rho > 0$ above.

So if $f'(z) \neq 0$ then f is locally a rotation followed by a scaling. In particular, see Figure 2.2, conformal mappings preserve the orientation and size of angles: two arcs ending at a point z and making an angle of θ are mapped to two arcs ending at the point $f(z)$ and make the same angle θ.

Conformal mappings are often used in applications of maths to study important but difficult to describe sets. For example the domain in Figure 2.3 is the Joukowsky aerofoil. It is the image of a disk under the mapping

$$f(z) = z + \frac{1}{z}.$$

A number of important partial differential equations are invariant under conformal mappings. This means that if U is a solution to a PDE defined on a set Ω_1 then $U \circ f$ is a solution to the same PDE on a domain Ω_2. Here

$$f \colon \Omega_2 \to \Omega_1$$

is a conformal mapping. Conformal mappings are also interesting in their own right and considerable work has been done on understanding the link between the shape of a domain and the mapping properties of conformal mappings of that domain.

2.4 Power series

A power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

for $a \in \mathbb{C}$ and a sequence of complex numbers $(a_n)_{n=0}^{\infty}$. The complex number a is called the centre of the power series. The radius of convergence of a power series, R, is defined by the relationship

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n},$$

where R can be 0 or ∞, when the upper limit on the right-hand-side is ∞ or 0 respectively.

You first see infinite series like this when you are introduced to Taylor series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

A Taylor series is a particular example of a power series with

$$a_n = \frac{f^{(n)}(a)}{n!},$$

but power series are more general than these.

Example 10. Consider the power series

$$\sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

The terms of this series are 0 for odd values of n, for even values they alternate between ± 1. So we have

$$|a_n| = \begin{cases} 1, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases}$$

Therefore

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1$$

and the power series converges for $|z| < 1$.

In real analysis it is not always clear why certain power series have finite radii of convergence. The example above highlights this. The power series is in fact the Taylor series

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n} = \frac{1}{1+z^2}.$$

The real-valued function $1/(1+x^2)$ is defined for all $x \in \mathbb{R}$ and is infinitely differentiable there, so it is not clear why the Taylor series does not converge for $|x| \geq 1$. The graph of $1/(1+x^2)$ is shown in Figure 2.4.

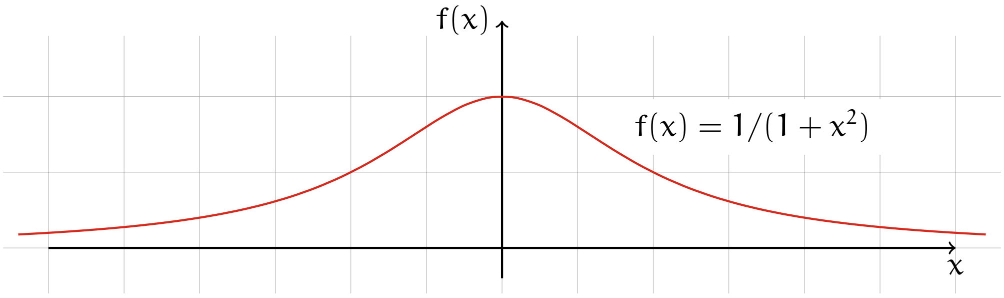


Figure 2.4: The graph of $f(x) = 1/(1+x^2)$.

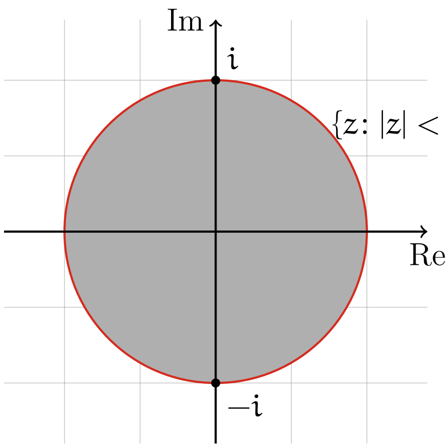


Figure 2.5: The radius of convergence for $f(z) = 1/(1+z^2)$.

The answer lies in considering this as a complex-valued function:

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}.$$

It is clear that this function is not defined at the points $z = \pm i$ and these lie on the boundary of the disk $\{z\colon |z| < 1\}$, see Figure 2.5. It it for this reason that the power series fails to converge for $|z| \geq 1$.

Theorem 2.10. *Let*

$$\begin{array}{rcl} f : B(a,R) & \to & \mathbb{C} \\ z & \mapsto & \displaystyle\sum_{n=0}^{\infty} a_n(z-a)^n \end{array}$$

where $R > 0$ *is the radius of convergence of the power series. Then*

$$\sum_{n=1}^{\infty} na_n(z-a)^{n-1} \tag{2.9}$$

also has radius of convergence R *and*

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1} \qquad |z| < R.$$

To prove this requires some manipulation of binomial series. We will use the following estimate of binomial coefficients.

Lemma 2.11. *For* $n \geq 2$ *and* $r = 0, 1, \ldots, n$,

$$\binom{n}{r+2} \leq n^2 \binom{n-2}{r}.$$

Proof. We have

$$\binom{n}{r+2} = \frac{n!}{(n-r-2)!(r+2)!}, \qquad \binom{n-2}{r} = \frac{(n-2)!}{(n-r-2)!r!}.$$

Therefore

$$\begin{aligned}\binom{n}{r+2}\Big/\binom{n-2}{r} &= \frac{n!}{(n-2)!}\frac{r!}{(r+2)!} \\ &= \frac{n(n-1)}{(r+2)(r+1)} \\ &\leq n(n-1) \leq n^2.\end{aligned}$$

□

Proof of Theorem 2.10. We first show that the radius of convergence of the power series (2.9) is R. Rewrite this power series in the standard form:

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}(z-a)^n.$$

First, suppose that $0 < R < \infty$, then

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} \in (0, \infty),$$

and we need to show that
$$\limsup_{n\to\infty}\Big((n+1)|a_{n+1}|\Big)^{1/n}=\frac{1}{R}.$$
Write the terms as follows:
$$\begin{aligned}\Big((n+1)|a_{n+1}|\Big)^{1/n}&=\exp\Big(\log\left((n+1)|a_{n+1}|\right)^{1/n}\Big)\\&=\exp\left(\frac{1}{n}\log(n+1)+\frac{1}{n}\log|a_{n+1}|\right)\\&=\exp\left(\frac{1}{n}\log(n+1)+\frac{n+1}{n}\log|a_{n+1}|^{1/(n+1)}\right).\end{aligned}$$
Then
$$\begin{aligned}\limsup_{n\to\infty}\left((n+1)|a_{n+1}|\right)^{1/(n+1)}&=\limsup_{n\to\infty}\exp\left(\frac{1}{n}\log(n+1)+\frac{n+1}{n}\log|a_{n+1}|^{1/(n+1)}\right)\\&=\exp(0+\log 1/R)=\frac{1}{R}\end{aligned}$$
as required. If $R=\infty$ then the same calculation holds although the last limit tends to 0 since
$$\limsup_{n\to\infty}|a_{n+1}|^{1/(n+1)}=0.$$

Next we will show that $g(z)=\sum_{n=1}^{\infty}na_n(z-a)^{n-1}$ is the derivative of $f(z)$. To simplify notation we assume without loss of generality that $a=0$, otherwise we can simply consider the functions $g(z+a)$ and $f(z+a)$. Let
$$E(h)=f(z+h)-f(z)-g(z)h$$
then by Theorem 2.2 we need to show that $E(h)/h\to 0$ as $h\to 0$. Now
$$\begin{aligned}E(h)&=f(z+h)-f(z)-g(z)h\\&=\sum_{n=0}^{\infty}a_n(z+h)^n-\sum_{n=0}^{\infty}a_nz^n-h\sum_{n=1}^{\infty}na_nz^{n-1}\\&=\sum_{n=0}^{\infty}a_n\Big((z+h)^n-z^n-nz^{n-1}h\Big)\\&=\sum_{n=0}^{\infty}a_nd_n(h)\end{aligned}$$
where
$$d_n(h)=(z+h)^n-z^n-nz^{n-1}h.$$

In order to show that $E(h)/h \to 0$ as $h \to 0$ we want to estimate the size of $|a_n|$ and the size of $|d_n(h)|$.

Now $d_0(h) = 0$ and $d_1(h) = 0$ for all h. For $n \geq 2$ the binomial theorem tells us

$$(z+h)^n = \sum_{k=0}^{n} \binom{n}{k} h^k z^{n-k} = z^n + nz^{n-1}h + \sum_{k=2}^{n} \binom{n}{k} h^k z^{n-k},$$

and therefore

$$d_n(h) = \sum_{k=2}^{n} \binom{n}{k} h^k z^{n-k}, \qquad n \geq 2.$$

Choose h with $|h|$ small enough so that we may find r_1 and r_2 with

$$|z| + |h| < r_1 < r_2 < R.$$

Now $r_2 \in B(0,R)$ and so the power series $f(z)$ converges for $z = r_2$. That is $\sum_{n=0}^{\infty} a_n r_2^n$ converges. It follows that $a_n r_2^n \to 0$ as $n \to \infty$, and the sequence $(a_n r_2^n)$ is bounded. Hence we may find a $M > 0$ such that $|a_n| r_2^n \leq M$, or

$$|a_n| \leq \frac{M}{r_2^n} \qquad \forall n.$$

Now, using the estimate in Lemma 2.11,

$$\begin{aligned} |d_n(h)| &= \left| \sum_{k=2}^{n} \binom{n}{k} h^k z^{n-k} \right| \\ &\leq \sum_{k=2}^{n} \binom{n}{k} |h|^k |z|^{n-k} \\ &= \sum_{i=0}^{n-2} \binom{n}{i+2} |h|^{i+2} |z|^{n-2-i}, \end{aligned}$$

where we are changing the indexing in the sum, $k = i + 2$,

$$\begin{aligned} &= |h|^2 \sum_{i=0}^{n-2} \binom{n}{i+2} |h|^i |z|^{n-2-i} \\ &\leq |h|^2 \sum_{i=0}^{n-2} n^2 \binom{n-2}{i} |h|^i |z|^{n-2-i}, \end{aligned}$$

by Lemma 2.11,

$$= |h|^2 n^2 (|h| + |z|)^{n-2}.$$

Since $|z| + |h| < r_1$ we can estimate this by

$$|d_n(h)| \leq |h|^2 n^2 r_1^{n-2}$$

and so using this and our estimate for $|a_n|$ we have that

$$\begin{aligned}
|E(h)| &= \left| \sum_{n=0}^{\infty} a_n d_n(h) \right| \\
&\leq \sum_{n=0}^{\infty} |a_n||d_n(h)| \\
&\leq \sum_{n=2}^{\infty} \frac{M}{r_2^n} n^2 |h|^2 r_1^{n-2} \\
&= M|h|^2 \sum_{n=2}^{\infty} n^2 \frac{r_1^{n-2}}{r_2^n}.
\end{aligned}$$

The sum on the right hand side converges by the ratio test. Therefore we have shown that $|E(h)| \leq C|h|^2$ for some constant $C > 0$. Hence $E(h)/h \to 0$ as $h \to 0$. By Theorem 2.2 we have shown that

$$f'(z) = g(z)$$

as required. □

This result highlights the importance of power series as a way of defining functions. This leads us to highlight the following class of functions.

Definition 2.12 – Analytic functions

A function on a domain Ω that can be written as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$$

with positive radius of convergence, $R > 0$, at each point $a \in \Omega$ is called an *analytic function*. The set of analytic functions on a domain Ω is written $\mathcal{A}(\Omega)$.

An analytic function is differentiable at each point in Ω and so we have $\mathcal{A}(\Omega) \subset \mathcal{H}(\Omega)$. We will show, eventually,

$$\mathcal{A}(\Omega) = \mathcal{H}(\Omega)$$

but in order to do that we will need to develop complex integration.

Corollary 2.13. *Suppose that* $f \in \mathcal{A}(\Omega)$ *then* $f^{(k)} \in \mathcal{A}(\Omega)$ *for each* $k \geq 0$.

This follows directly from the theorem since, if $\sum_{n=0}^{\infty} a_n(z-a)^n$ is analytic then so is its derivative $\sum_{n=0}^{\infty} na_n(z-a)^{n-1}$, therefore so is its derivative and so on. Furthermore the radius of convergence is the same for each derivative. In fact it can be seen from the theorem that the kth derivative is

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} a_{n+k}(z-a)^n. \tag{2.10}$$

Letting $z = a$ in this equation gives

$$f^{(k)}(a) = k! a_k,$$

and so if $f \in \mathcal{A}(\Omega)$ then at each point $a \in \Omega$ it is equal to its Taylor series centred at a for some disk $B(a, r)$. We state this as the following corollary.

Corollary 2.14. *Suppose that* $f \in \mathcal{A}(\Omega)$ *and* $a \in \Omega$ *then there is a* $r > 0$ *such that*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

for all $z \in B(a, r)$.

2.5 The exponential function

Power series are much more flexible than standard function definitions. This means we can use them to define properly many of the standard functions you encounter in analysis. The exponential function in real analysis is the function

$$\begin{array}{rccc} \exp & : \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & e^x \end{array}.$$

This definition makes sense when $x \in \mathbb{Q}$ since $\exp p/q$ can be defined as the qth root of e^p. Or similarly the smallest positive real solution to the equation

$$y^q - e^p = 0.$$

The meaning of the expression e^x is far less clear when x is irrational. However given the development of the theory of power series in the last section we can define a function exp as follows.

$$\begin{array}{rccc} \exp & : \mathbb{C} & \to & \mathbb{C} \\ & z & \mapsto & \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}. \end{array}$$

This definition makes sense for all $z \in \mathbb{C}$ since the radius of convergence R, satisfies

$$\frac{1}{R} = \limsup_{n\to\infty} |n!|^{-1/n} = 0$$

and so $R = \infty$. By Theorem 2.10 exp is differentiable everywhere and hence is holomorphic in $\mathbb{C}$. This is therefore an entire function.

Proposition 2.15. *The exponential function satisfies the following:*

1. $\exp(z+w) = \exp z \exp w$.
2. $\dfrac{1}{\exp z} = \exp(-z)$.
3. $\exp(z + 2\pi i) = \exp z$
4. $\dfrac{d}{dz}\exp z = \exp z$

Lemma 2.16 (The Cauchy product)**.** *Suppose that* $\sum_{i=0}^{\infty} a_i$ *and* $\sum_{j=0}^{\infty} b_j$ *are two absolutely convergent complex series. Then*

$$\left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right) = \sum_{k=0}^{\infty} c_k$$

where

$$c_k = \sum_{l=0}^{k} a_l b_{k-l}$$

Proof of Theorem 2.15. 1. By the Binomial theorem

$$\frac{1}{n!}(z+w)^n = \frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m} z^m w^{n-m} = \sum_{k=0}^{n}\frac{z^m w^{n-m}}{m!(n-m)!}.$$

So

$$\exp(z+w) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{z^m w^{n-m}}{m!(n-m)!}. \tag{2.11}$$

On the other hand, using the Cauchy product and the fact that the exponential series is entire and hence converges absolutely for all values,

$$\begin{aligned}\exp z \exp w &= \left(\sum_{i=0}^{\infty}\frac{z^i}{i!}\right)\left(\sum_{j=0}^{\infty}\frac{z^j}{j!}\right)\\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{z^l}{l!}\frac{w^{k-l}}{(k-l)!}.\end{aligned}$$

Comparing this series with (2.11) proves that

$$\exp(z + w) = \exp z \exp w$$

as required.

2. This follows from part 1 with $w = -z$, since $\exp 0 = 1$.

3. By part 1 we have that

$$\exp(z + 2\pi i) = \exp z \exp(2\pi i).$$

However by Euler's formula $\exp 2\pi i = \cos 2\pi + i \sin 2\pi = 1$ (this needs to be proved as well but we can only do so after we have defined $\sin$ and $\cos$ for complex values).

4. Finally given Theorem 2.10 we have that

$$\begin{aligned} \frac{d}{dz} \exp z &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{n z^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} \\ &= \exp z. \end{aligned}$$

□

Note that since $\exp(a + ib) = \exp(a) \exp(ib)$ we have the useful identity.

$$|\exp z| = \exp \operatorname{Re}(z).$$

2.6 *The logarithm*

The natural way to define the logarithm is to define it as the inverse function of the exponential function, $\exp$. In order to do that though we need $\exp$ to be bijective and this is not true in general. From Theorem 2.15 part 3, for any $k \in \mathbb{Z}$

$$\exp z = \exp(z + 2k\pi i).$$

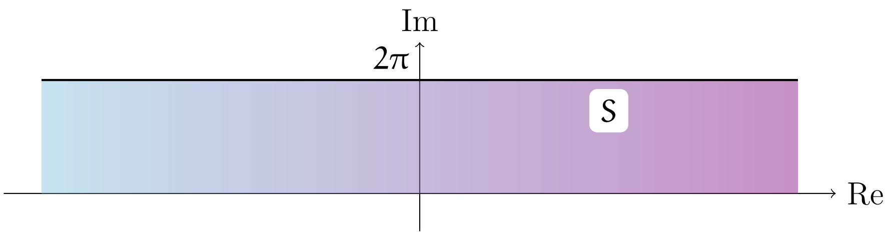


Figure 2.6: The strip S in Theorem 2.17

Therefore exp is not injective. This is similar to the argument and we treat it in the same way: we restrict the possible values to numbers $z \in \mathbb{C}$ with $\operatorname{Im} z \in [0, 2\pi)$. Given any $w \in \mathbb{C}$ we can choose a suitable k and a number $z \in \mathbb{C}$ with $\operatorname{Im} z \in [0, 2\pi)$ so that

$$\exp z = \exp w.$$

This is used to prove the following, see Figure 2.6.

Theorem 2.17. *Let* $S = \{z \in \mathbb{C} \colon 0 \le \operatorname{Im} z < 2\pi\}$ *then* exp *is a bijective function from* S *to* $\mathbb{C}\backslash\{0\}$*.*

Proof. Suppose that $\exp z = \exp w$ for $z, w \in S$. Then $z = w + 2k\pi i$ for some $k \in \mathbb{Z}$. Since $z, w \in S$ we have that

$$0 \le \operatorname{Im} z < 2\pi, \qquad 0 \le \operatorname{Im} w < 2\pi$$

and so

$$-2\pi < \operatorname{Im}(z - w) < 2\pi.$$

Therefore if $\operatorname{Im}(z - w) = 2k\pi$ then $k = 0$ and $z = w$. Hence exp is injective.

To prove that it is surjective take any $w \in \mathbb{C}\backslash\{0\}$. Let $r = |w|$ and $\theta = \arg w$ then we may choose $\theta \in [0, 2\pi)$. Letting $z = \log|w| + i\theta$ we have that

$$\begin{aligned} \exp z &= \exp(\log|w| + i\theta) \\ &= \exp(\log|w|)\exp(i\theta) \\ &= |w|\exp(i\theta) = w \end{aligned}$$

and so exp is surjective. □

The inverse of exp defined in the theorem above is called the *principle logarithm*:

$$\operatorname{Log} w = \log|w| + i \operatorname{Arg} w \tag{2.12}$$

where Arg represents the principle argument of w, i.e. the argument contained in $[0, 2\pi)$.

Example 11. The principle log of $1+i$.

Here $|1+i| = \sqrt{2}$ and the $\mathrm{Arg}(1+i) = \pi/4$ so

$$\mathrm{Log}(1+i) = \frac{1}{2}\log 2 + i\frac{\pi}{4}.$$

An important point to note here is that whereas $\log x$ is not defined in real analysis when $x < 0$ it is possible to define it in complex analysis.

Example 12. The principle log of -2.

Here we can write $-2 = 2\exp(i\pi)$. Then

$$\mathrm{Log}(-2) = \log 2 + i\pi.$$

Some books consider the principle argument to consist of arguments in the interval $[-\pi, \pi)$. There is no substantive difference in doing this. In fact our choice of strip S in the theorem and hence the choice of interval is arbitrary. The proof can be modified quite easily to show the following.

Corollary 2.18. *Suppose that* $\alpha \in \mathbb{R}$. *Let* $S_\alpha = \{z \in \mathbb{C}\colon \alpha \le \mathrm{Im}\, z < \alpha + 2\pi\}$. *Then* $\exp$ *is a bijective mapping from* S_α *to* $\mathbb{C}\backslash\{0\}$.

We define the inverse of $\exp\colon S_\alpha \to \mathbb{C}\backslash\{0\}$ as $\log_\alpha$, the α-logarithm. The function $\log_\alpha$ is called a *branch of the logarithm*.

The logarithm is not holomorphic everywhere, there is a problem when the argument changes from α to $\alpha + 2\pi$. At this point it is no longer continuous and therefore cannot be differentiable. It is holomorphic everywhere else though.

Theorem 2.19. *The* α*-logarithm is holomorphic on* $\mathbb{C}\backslash\{re^{i\alpha}\colon r \ge 0\}$. *Its derivative is* $1/z$ *there.*

Proof. We will simply verify the polar form of the Cauchy-Riemann equations, (2.8). Since

$$\mathbb{C}\backslash\{re^{i\alpha}\colon r \ge 0\}$$

is open this will suffice by Theorem 2.5.

This is straightforward though since $u(r,\theta) = \log r$ and $v(r,\theta) = \theta$, where $\theta \in [\alpha, \alpha + 2\pi)$. It follows that

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \qquad \frac{\partial u}{\partial \theta} = 0,$$
$$\frac{\partial v}{\partial r} = 0, \qquad \frac{\partial v}{\partial \theta} = 1.$$

Therefore
$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta},$$
and
$$\frac{\partial u}{\partial \theta} = 0 = -\frac{1}{r}\frac{\partial v}{\partial r}.$$
Therefore these satisfy the polar form of the Cauchy-Riemann equations, (2.8), and so by Theorem 2.5 $\log_\alpha$ is holomorphic. To find the derivative simply note that

$$\exp(\log_\alpha z) = z$$

for any α, we can then use the chain rule and inverse function theorem. The left-hand-side becomes

$$\exp(\log_\alpha z)\frac{d}{dz}(\log_\alpha z) = z\frac{d}{dz}\log_\alpha z$$

and the right-hand-side becomes 1. □

2.7 Complex powers

Suppose that $\alpha \in \mathbb{C}$ then we define

$$z^\alpha = \exp(\alpha \log z).$$

If α is an integer then this has its usual meaning (multiplying z by itself α times). The function $z \mapsto z^\alpha$, in this case, is said to be *single-valued* since there is only one possible value. If $\alpha = p/q$ for $p, q \in \mathbb{Z}$ ($q \neq 0$) then $z^{p/q}$ has q distinct values and the function $z \mapsto z^\alpha$ is said to be q-valued or just multiple-valued. This should be familiar from the real-valued function $x \mapsto \sqrt{x}$. You have learned that $\sqrt{x}$ means the positive square root, however the square root is in reality either positive or negative. When x is real we can simplify this by always using the positive square root. This is no longer possible in complex analysis since there is no way of preferring one root over another.

This may seem like a problem but is not in reality. Riemann considered this in great depth and realised that it could be overcome if instead of treating the roots as different numbers, treating them as the *same* number. So that the $z \mapsto z^\alpha$ maps one value to one of these abstracted values. This led to the notion of a *Riemann surface* which is a generalisation of the complex plane. We will not go through this but it would make a good topic for a project.

For other values of α the function is infinite-valued.

2.8 Trigonometric functions

We can also define trigonometric functions using power series.

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

These functions are also entire functions and therefore we can rearrange terms without hesitation to deduce the identities

$$\cos z = \frac{\exp(iz) + \exp(-iz)}{2}, \qquad \text{and} \qquad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i}.$$

Proposition 2.20. *The following identities hold*

1. $\exp iz = \cos z + i \sin z$ *for* $z \in \mathbb{C}$.
2. $\cos^2 z + \sin^2 z = 1$
3. $\frac{d}{dz} \cos z = -\sin z$, *and* $\frac{d}{dz} \sin z = \cos z$

Proof. The proof is left as an exercise. □

Things to know

- The definition of complex differentiation;
- The product, chain and quotient rules;
- The definition of a holomorphic function, and simple examples;
- The statement of Theorem 2.2;
- The Cauchy-Riemann equations;
- The definition of a power series and how to calculate its radius of convergence;
- Understand the statement of Theorem 2.10;
- The definition of an analytic function;
- The definitions of the standard functions as power series;
- The logarithm and branches of logarithms.

Problems

1. Find the radius of convergence for the following power series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^k}{a^n} z^n$ for $a \in \mathbb{C}$;

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n - 1}{n} z^n$

2. Suppose that the sequence $(|a_n|)$ is decreasing. Show that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence *at least* 1

3. Prove the following:

   (a) $\exp iz = \cos z + i \sin z$ for $z \in \mathbb{C}$;

   (b) $\cos^2 z + \sin^2 z = 1$;

   (c) $\dfrac{d}{dz} \cos z = -\sin z$, and $\dfrac{d}{dz} \sin z = \cos z$.

4. Show that $\operatorname{Log}(zw) = \operatorname{Log} z + \operatorname{Log} w$ whenever $\operatorname{Im} z > 0$ and $\operatorname{Im} w > 0$. Give an example of points $z, w \in \mathbb{C}$ such that $\operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w$.

5. Describe a set on which we can define $\log(z - i)$

6. Suppose that $p(z)$ is a polynomial of degree n. Explain why p is analytic in $\mathbb{C}$.

7. Suppose that $p(z) = \prod_{i=1}^{n} (z - \alpha_i)$ is a polynomial. Prove that

$$\frac{p'(z)}{p(z)} = \sum_{i=1}^{n} \frac{1}{z - \alpha_i}.$$

Chapter 3 : Complex Integration

“Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.”

W. S. Anglin, *Mathematics and History*, Mathematical Intelligencer v. 4, no. 4

3.1 Integration of complex functions

We begin with the familiar notion of integration. Suppose that $f(t)$ is a complex-valued function on an interval (a, b). We write this as

$$f(t) = u(t) + iv(t)$$

then we can define the integral of f (assuming it exists) as

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

This integral inherits a number of properties from the real-integral counterparts. We will collect two here for use elsewhere.

Proposition 3.1. *Let* $f(t) = u(t) + iv(t)$ *be a complex-valued function on an interval* (a, b) *such that both* u *and* v *are integrable.*

1. *For* $c \in \mathbb{C}$,
$$\int_a^b cf(t)dt = c\int_a^b f(t)dt;$$
2. *The triangle inequality holds,*
$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt.$$

Proof. The first property follows on writing $c = d + ie$ and expanding the integral. To prove the second property assume that the integral is non-zero. Let

$$\theta = \arg\int_a^b f(t)dt$$

then, with $c = e^{-i\theta}$, we have that

$$\begin{aligned} \left| \int_a^b f(t)\,dt \right| &= \mathrm{Re}\left(e^{-i\theta} \int_a^b f(t)\,dt \right) \\ &= \int_a^b \mathrm{Re}\left(e^{-i\theta} f(t) \right) dt \\ &\leq \int_a^b |f(t)|dt \end{aligned}$$

Since $\mathrm{Re}(e^{-i\theta}f(t)) \leq |e^{-i\theta}f(t)| = |f(t)|$. □

3.2 *Integration along curves*

Recall the following definitions from calculus.

Definition 3.2 – Curves and paths

- A *path* is a continuous function $p\colon [a, b] \to \mathbb{C}$;
- The *trace* of a path is the set $\mathbf{tr}\, p = \{z \in \mathbb{C} \colon z = p(t), t \in [a, b]\}$;
- A set $C \in \mathbb{C}$ is a *curve* if it is the trace of some path.

We will always assume that a path p is piecewise continuously differentiable. If $\mathbf{tr}\, p = C$ then we say C is parametrised by p.

For example the curve C in $\mathbb{R}^2$ parametrised by

$$p(t) = (\sin 2t, \cos t)^T \qquad t \in [0, \pi]$$

is drawn in Figure 3.1. We normally include an arrow on a curve to represent its orientation. In this example $t = 0$ is mapped to the top of the contour $(0, 1)^T$, as t increases the parametrisation represents points moving along the curve heading towards the point at the bottom, $(0, -1)^T$, which corresponds to $t = \pi$.

In this section we will develop, in the same way that you did in real analysis, the technique of finding the integral of a complex-valued function along a curve. In the complex plane we have the advantage of being able to define a curve as a function of complex numbers. For example the curve C of Figure 3.1 could just as easily have been defined as

$$p(t) = \sin 2t + i \cos t, \qquad t \in [0, \pi].$$

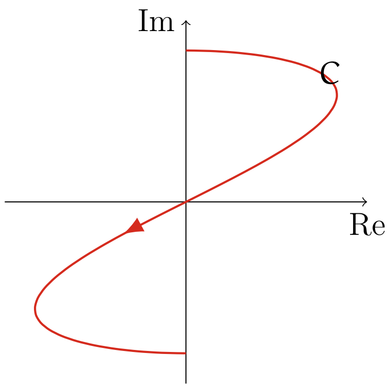


Figure 3.1: The curve $C = \{(\sin 2t, \cos t)^T : t \in [0, \pi]\}$

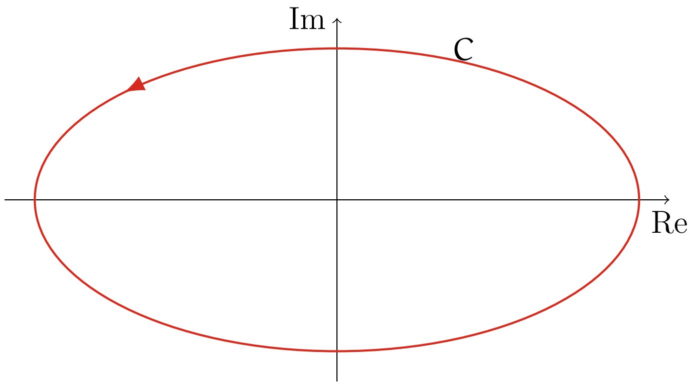


Figure 3.2: The curve $C = \{2\cos t + i \sin t : t \in [0, 2\pi]\}$

Example 13. The curve C parametrised by

$$p(t) = 2\cos t + i \sin t, \qquad t \in [0, 2\pi]$$

is an ellipse with major axis 2 and minor axis 1. See Figure 3.2.

For a piecewise continuously differentiable curve C of finite length we can define the integral of a continuous function f along C as the limit of Riemann sums in the following way. Let P be a partition of C, i.e. suppose that

$$P = \{z_0, z_1, \ldots, z_n\}$$

where z_0 and z_n are the endpoints of C. Then we let

$$S(f,P) = \sum_{k=0}^{n-1} f(z_k)(z_{k+1} - z_k).$$

Let $I \in \mathbb{C}$. If for each $\epsilon > 0$ there is a partition P such that

$$|S(f,P) - I| < \epsilon$$

then we say I is the integral along C of f and write

$$\int_C f(z)dz = I.$$

In general we don't need this limiting argument, since we can define the integral using a parametrisation $p(t)$ as follows.

Definition 3.3 – Integral along a curve

Let $p(t)$, $a \leq t \leq b$, be a path. Suppose that $p'(t)$ exists almost everywhere and f is a complex-valued function such that $f(p(t))$ is integrable. Then

$$\int_C f(z)dz = \int_a^b f(p(t))p'(t)dt.$$

Example 14. Suppose that $f(z) = 2z$ and C is parametrised by $p(t) = t + t^2 i$ for $-1 \leq t \leq 1$.

Here $f(p(t)) = 2(t + t^2 i) = 2t + 2t^2 i$, and $p'(t) = 1 + 2ti$. Therefore

$$\begin{aligned}
\int_C f(z)dz = \int_{-1}^{1} f(p(t))p'(t)dt &= \int_{-1}^{1} (2t + 2t^2 i)(1 + 2ti)dt \\
&= \int_{-1}^{1} -4t^3 + 2t + 6it^2 dt \\
&= \int_{-1}^{1} -4t^3 + 2t dt + i\int_{-1}^{1} 6t^2 dt \\
&= 4i.
\end{aligned}$$

When you learned about integrals along curves in calculus a number of important properties were proved that have direct analogues in complex integration. Firstly if $C = C_1 \cup C_2$ then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

For any curve C defined by $p(t)$, $a \le t \le b$, we may define the curve $-C$ as the same curve but in the opposite direction. So $-C$ is parametrised by

$$q(t) = p(-t) \qquad -b \le t \le -a.$$

Then

$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

We won't prove these. Instead we prove the following that relates the complex integral along a curve to the real integral along a curve that you learned about in calculus.

Proposition 3.4. *Suppose that* $f\colon \mathbb{C} \to \mathbb{C}$*. Write*

$$f(x+iy) = u(x,y) + iv(x,y).$$

Let $\boldsymbol{a}, \boldsymbol{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be vector fields defined by*

$$\boldsymbol{a}(x,y) = \begin{pmatrix} u & -v \end{pmatrix}^T;$$
$$\boldsymbol{b}(x,y) = \begin{pmatrix} v & u \end{pmatrix}^T,$$

and suppose that C *is a curve. Then*

$$\int_C f(z)dz = \int_C \boldsymbol{a} \cdot d\boldsymbol{r} + i\int_C \boldsymbol{b} \cdot d\boldsymbol{r}$$

Proof. Suppose that $p\colon [c,d] \to \mathbb{C}$ is a path parametrising C. Then write $p(t) = p_1(t) + ip_2(t)$.

$$\begin{aligned}
\int_C f(z)dz &= \int_c^d f(p(t))p'(t)dt \\
&= \int_c^d \Big(\big(u(p(t)) + iv(b(t))\big)\big(p_1'(t) + ip_2'(t)\big)\Big)dt \\
&= \int_c^d \big(u(p(t))p_1'(t) - v(p(t))p_2'(t) + iv(p(t))p_1'(t) + iu(p(t))p_2'(t)\big)dt \\
&= \int_c^d \big(u(p(t))p_1'(t) - v(p(t))p_2'(t)\big)dt + i\int_c^d \big(v(p(t))p_1'(t) + u(p(t))p_2'(t)\big)dt \\
&= \int_c^d \mathbf{a}(p(t)) \cdot p'(t)dt + i\int_c^d \mathbf{b}(p(t)) \cdot p'(t)dt \\
&= \int_C \mathbf{a} \cdot d\mathbf{r} + i\int_C \mathbf{b} \cdot d\mathbf{r}.
\end{aligned}$$

□

The purpose of this proposition is so that we may use the previously proved results on integration of vector fields along curves in the definition of a complex integral over a curve. For example as a consequence of this result that two different parameterisations of the curve C lead to the same value for the integral, since this was proved for vector fields.

3.3 Integration with respect to arc-length

Recall that the length of a curve C, parametrised by $p(t)$, $a \leq t \leq b$, is given by the formula

$$\text{length}(C) = \int_a^b |p'(t)| dt.$$

For example the circle C_r of radius r can be defined by $p(t) = re^{it}$ for $0 \leq t \leq 2\pi$. The length of C_r is

$$\begin{aligned}\text{length}(C_r) &= \int_0^{2\pi} |ire^{it}| dt = \int_0^{2\pi} r dt \\ &= 2\pi r.\end{aligned}$$

This is an example of an integral with respect to arc-length. In general we have the following definition.

Definition 3.5 – Integral with respect to arc-length

Suppose that C is a piecewise continuously differentiable curve with parametrisation $p(t)$, $a \leq t \leq b$. The integral with respect to arc-length of an integrable function f is

$$\int_C f(z)|dz| = \int_a^b f(p(t))|p'(t)| dt.$$

So we could write

$$\text{length}(C) = \int_C |dz|.$$

Integration with respect to arc-length is different to integration along a curve, and is useful in complex analysis since it provides a triangle inequality for integrals.

Proposition 3.6. *Suppose that f is a complex-valued function defined on a piecewise continuously differentiable curve C. Then the following inequality holds.*

$$\left| \int_C f(z) dz \right| \leq \int_C |f(z)| |dz|.$$

Proof. Using the triangle inequality, Proposition 3.1 part 2, we have that

$$\begin{aligned}\left|\int_C f(z)dz\right| &= \left|\int_a^b f(p(t))p'(t)dt\right| \\ &\le \int_a^b |f(p(t))|\,|p'(t)|dt \\ &= \int_C |f(z)|\,|dz|.\end{aligned}$$

□

You should make sure you understand what each of these quantities means.

As a consequence we have the following useful estimate of the contour integral over an arc of finite length.

$$\left|\int_C f(z)dz\right| \le \text{length}(C) \sup_{z\in C} |f(z)|. \tag{3.1}$$

3.4 Some topological notions

Complex integration depends on curves. As such we need to collect together some ideas that help us describe correctly curves and other considerations.

First remember the following definitions.

Definition 3.7. Let C be a curve parameterised by $p(t)$ for $a \le t \le b$.

- If $p(a) = p(b)$ then C is called a *closed curve*.
- If p is injective then C is called a *simple curve*.

Informally a closed curve is one that loops round and joins up at its endpoints. A simple curve is one that does not cross itself. See Figure 3.3.

Theorem 3.8 – The Jordan Curve Theorem

Let C be a simple closed curve in $\mathbb{C}$. Then $\mathbb{C}\backslash C$ consists of two components, a bounded component and an unbounded component.

Drawing a diagram of this makes it look like a trivial result. Figure 3.4 does that. However it is not trivial and is in fact quite difficult to prove. A simple closed curve is often called a *Jordan curve* because of this theorem.

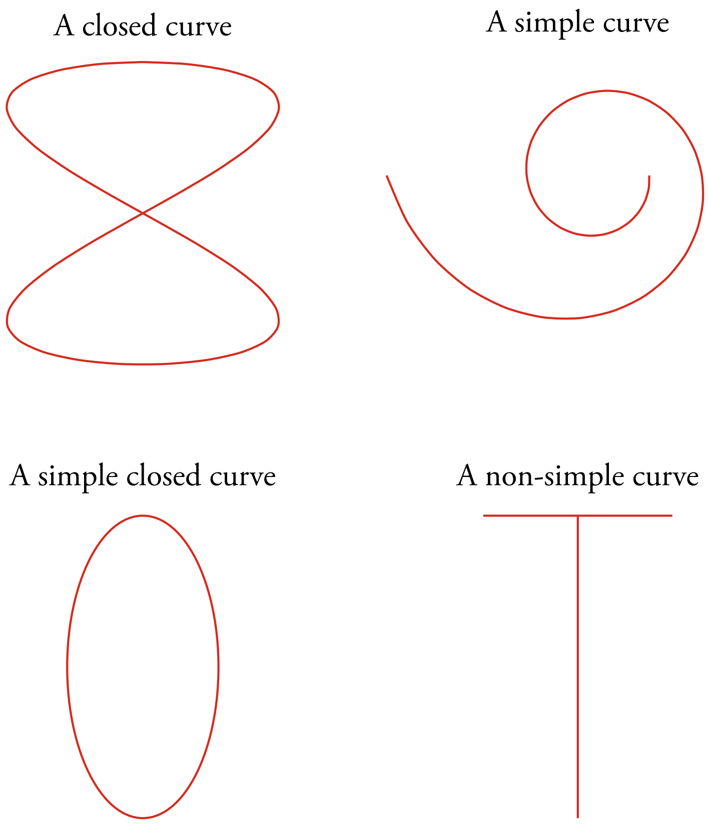


Figure 3.3: Types of curves

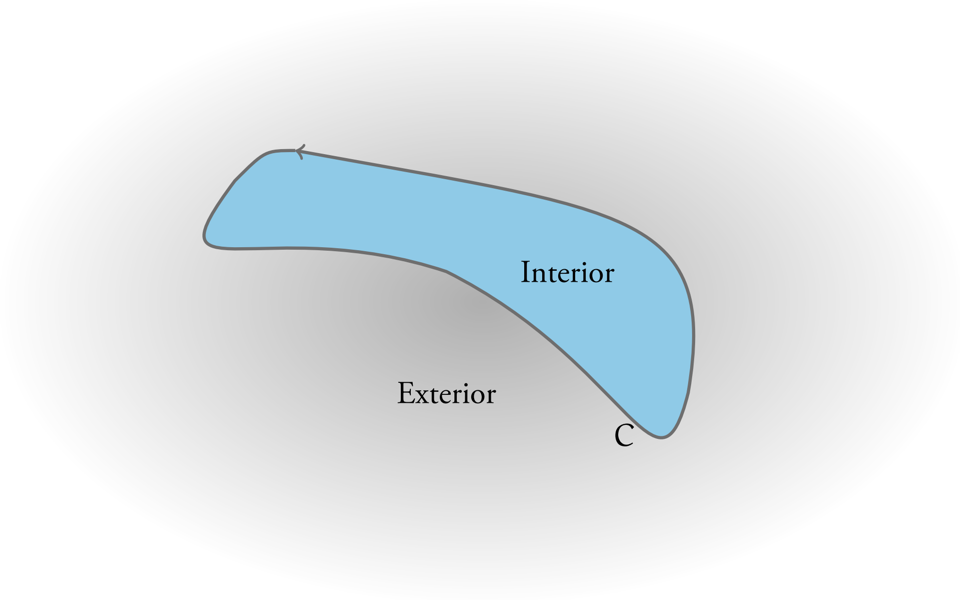


Figure 3.4: The Jordan curve theorem

Jordan curves need not be well-behaved like the one in Figure 3.4. The snowflake curve for example is a Jordan curve. It is possible for Jordan curves to have positive area, [5].

In the long tradition of courses in complex analysis I will not go through this proof, it's rather long and involves many interesting ideas. In [1] it is merely mentioned as a side-comment. The full proof can be found in §21.3 of [3, pages 661-667].

Previously we said that a *domain* is a set in $\mathbb{C}$ that is open and connected, see Definition 1.6. You spent a considerable time last year learning about open sets in metric spaces and these are quite familiar in $\mathbb{C}$. A set Ω is connected (also called path-connected) if for any two points $z, w \in \Omega$ there is a path $p\colon [0,1] \to \Omega$ with $p(0) = z$ and $p(1) = w$.

We will also make use of the following definition (Definition 5.6 from the calculus notes).

Definition 3.9 – Simply connected domains

A set $\Omega \subset \mathbb{C}$ is *simply connected* if

- Ω is path-connected;
- the interior of any closed curve $C \subset \Omega$ is contained in Ω.

Informally a simply connected domain is one without holes in it. Figure 3.5 shows two domains

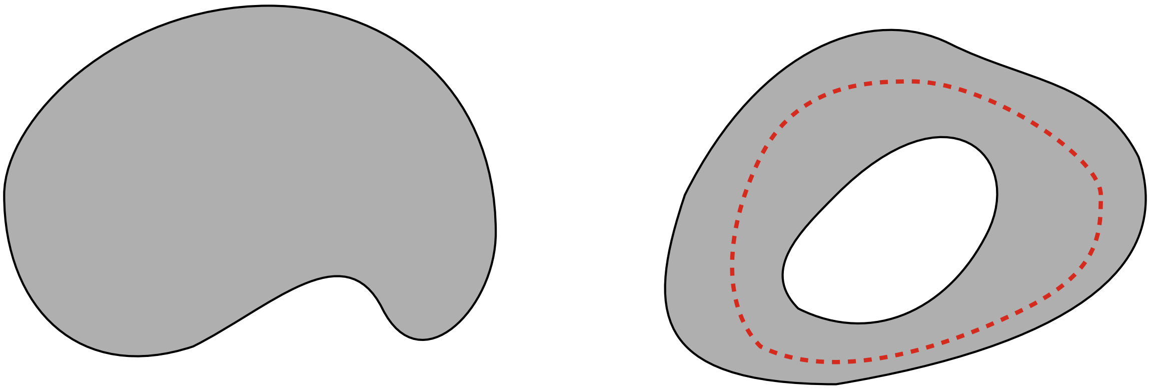


Figure 3.5: Simply and multiply connected domains

the first is simply connected and the second is not — a domain that is not simply connected is called *multiply connected*. The red dashed curve is a simple closed curve whose interior is not a subset of the domain.

3.5 Cauchy's theorem

It doesn't matter how we parameterise a curve, the integral of a function over that curve is the same. This follows from Proposition 3.4 and the equivalent result for vector fields. In this section we will show that under some restrictions on f it doesn't matter what the curve looks like either, as long as the end-points are the same.

This is one of the most important theorems in complex analysis. Under fairly mild assumptions of complex-differentiability and curves of finite length it turns out that integrating along a curve depends only on the endpoints.

The entirety of this section will be devoted to proving the following.

Theorem 3.10 – Cauchy's theorem

Suppose that $f(z)$ is holomorphic in the simply connected domain Ω. Then for any simple closed curve C in Ω of finite length

$$\int_C f(z)dz = 0.$$

In order to understand the importance of this theorem, consider a function $f(z)$ holomorphic in the domain Ω and suppose that a and b are two points in Ω. Let C_1 and C_2 be two curves

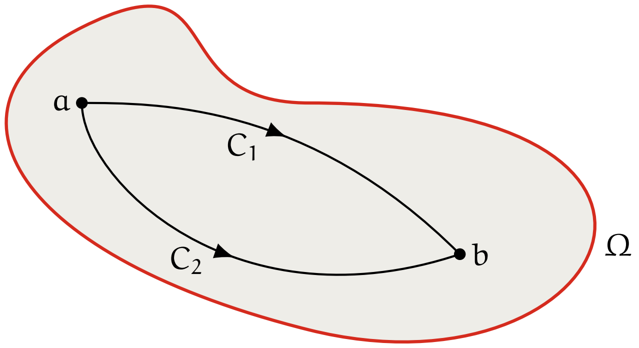


Figure 3.6: A consequence of Cauchy's theorem

joining a to b, see Figure 3.6. Then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

This follows because we can define the simple closed curve $C = C_1 \cup (-C_2)$. Then by Cauchy's theorem

$$\begin{aligned} 0 = \int_C f(z)dz &= \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz \\ &= \int_{C_1} f(z)dz - \int_{C_2} f(z)dz. \end{aligned}$$

This argument only follows if C_1 and C_2 do not intersect, but even if they do the conclusion is still true. In general as long as $f(z)$ is holomorphic on a simply connected domain containing both arcs the integrals will be the same.

3.6 The weak form of Cauchy's Theorem

It is possible to prove a weak form of Cauchy's theorem using the Cauchy-Riemann equations and Green's theorem, via Proposition 3.4 that we proved earlier. Since it is so much more simple we provide it here before moving on to our stronger form.

Theorem 3.11 – Weak form of Cauchy's theorem

Suppose that $f(z)$ is holomorphic in the simply connected domain Ω. Let $f(x+iy) = u(x,y) + iv(x,y)$ and suppose that u and v are continuously differentiable. Then for any simple closed curve C in Ω of finite length

$$\int_C f(z)dz = 0.$$

The difference is in the assumption of continuous derivatives in the hypothesis. This allows us to use Green's theorem. This is the form of the theorem that Cauchy proved in 1825. It wasn't until 1900 that the mathematician Goursat devised a method for dispensing with the requirement that the derivative be continuous.

Proof of the weak form of Cauchy's theorem. With the notation of Proposition 3.4 we need only show that

$$\int_C \mathbf{a} \cdot d\mathbf{r} = 0, \quad \text{and} \quad \int_C \mathbf{b} \cdot d\mathbf{r} = 0.$$

First $\mathbf{a} = \begin{pmatrix} u & -v \end{pmatrix}^T$ and so, since these functions are continuously differentiable, by Green's theorem

$$\int_C \mathbf{a} \cdot d\mathbf{r} = \iint_U \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy,$$

where U is the interior of C.

However, since f is holomorphic, the Cauchy-Riemann equations (2.4) hold and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The integrand on the right is therefore 0 throughout U and so the integral is 0.

Similarly, again using Green's theorem,

$$\int_C \mathbf{b} \cdot d\mathbf{r} = \iint_U \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy$$

and, again, by the Cauchy-Riemann equations the integral on the right is 0. □

3.7 *Proof of Cauchy's theorem*

We will prove Cauchy's theorem in three steps, the first has already been done by proving the weak form in the last section. The next step is to prove the theorem for the perimeter of a triangle. This

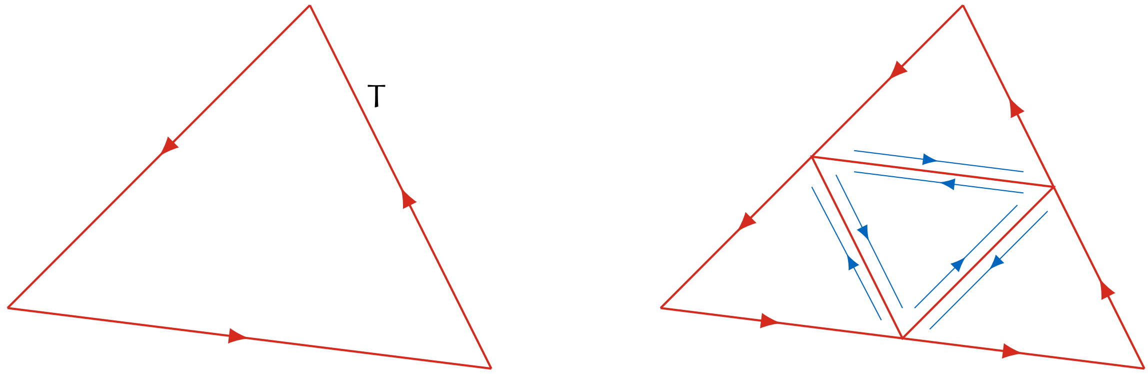


Figure 3.7: Breaking a triangle into four sub-triangles

is based on an idea of Pringsheim. Goursat originally proved the theorem with rectangles instead of triangles. I prefer this proof, based on [4], since it involves a direct proof without any additional sidesteps or excursions.

Theorem 3.12 – Cauchy's theorem for a triangle

Suppose that f is holomorphic in a simply connected domain Ω, and T is the perimeter of a triangle contained in Ω. Then

$$\int_T f(z)dz = 0.$$

Proof. We divide the triangle T into four similar triangles, T_1, T_2, T_3 and T_4 by joining the midpoints of each side of T as indicated in Figure 3.7. The common sides of the triangles cancel each other when integrating since they are oriented in opposite directions. So

$$\int_T f(z)dz = \int_{T_1} f(z)dz + \int_{T_2} f(z)dz + \int_{T_3} f(z)dz + \int_{T_4} f(z)dz.$$

By the triangle inequality we have that

$$\left|\int_T f(z)dz\right| \leq \left|\int_{T_1} f(z)dz\right| + \left|\int_{T_2} f(z)dz\right| + \left|\int_{T_3} f(z)dz\right| + \left|\int_{T_4} f(z)dz\right|.$$

Therefore at least one of the integrals on the right hand side satisfies

$$\left|\int_{T_i} f(z)dz\right| \geq \frac{1}{4}\left|\int_T f(z)dz\right|.$$

We rename T_i as S_1 and continue this process with this triangle: Subdivide the triangle S_1 into four similar sub-triangles, with side-length half of the side-length of S_1. It follows again that we may choose one of these smaller triangles, that we will call S_2, such that

$$\left|\int_{S_2} f(z)dz\right| \geq \frac{1}{4}\left|\int_{S_1} f(z)dz\right| \geq \frac{1}{4^2}\left|\int_T f(z)dz\right|.$$

We continue in this way finding a sequence of rectangles S_k, $k = 1, 2, \ldots$ each smaller than the last, with side-length halved each time, and satisfying

$$\left|\int_{S_k} f(z)dz\right| \geq \frac{1}{4^k}\left|\int_T f(z)dz\right|. \tag{3.2}$$

Let $\tilde{S}_k$ denote the union of S_k and its interior. We claim that the intersection of these triangles is a set containing a single point, z_0. To see this let us define

$$\operatorname{diam} A = \sup_{a,b\in A} |a - b|$$

to be the diameter of a set A. Since $\tilde{S}_k$ is a triangle $\operatorname{diam} \tilde{S}_k$ is the length of the largest side. Therefore we have that

$$\operatorname{diam} \tilde{S}_{k+1} = \frac{1}{2} \operatorname{diam} \tilde{S}_k, \qquad k = 1, 2, \ldots.$$

It follows by induction that

$$\operatorname{diam} \tilde{S}_k = \frac{1}{2^k} \operatorname{diam} T.$$

Therefore $\operatorname{diam} \tilde{S}_k \to 0$ as $k \to \infty$ and there is a z_0 such that

$$\cap_{k=1}^{\infty} \tilde{S}_k = \{z_0\}.$$

Now f is holomorphic at z_0 and so by Theorem 2.2 there is a function $E(h)$ such that

$$f(z_0 + h) = f(z_0) + f'(z_0)h + E(h)$$

and $E(h)/h \to 0$ as $h \to 0$. Write $z = z_0 + h$ then this becomes

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z - z_0).$$

Since $f(z_0)+f'(z_0)(z-z_0)$ is a linear polynomial it has continuous derivatives of all orders, therefore by the weak form of Cauchy's theorem we have that

$$\int_{S_k} (f(z_0) + f'(z_0)(z - z_0))dz = 0.$$

It follows that for each k

$$\begin{aligned}\int_{S_k} f(z)dz &= \int_{S_k} (f(z_0) + f'(z_0)(z - z_0))dz + \int_{S_k} E(z - z_0)dz \\ &= \int_{S_k} E(z - z_0)dz.\end{aligned}$$

Let $\epsilon > 0$. Since $E(z - z_0)/(z - z_0) \to 0$ as $z \to z_0$ we may find n so that for all $z \in \tilde{S}_n$

$$|E(z - z_0)|/|z - z_0| < \epsilon. \tag{3.3}$$

Hence

$$\begin{aligned}\left|\int_{S_k} f(z)dz\right| &= \left|\int_{S_k} E(z - z_0)dz\right| \\ &\le \int_{S_k} |E(z - z_0)||dz|,\end{aligned}$$

by Proposition 3.6,

$$\le \epsilon \int_{S_k} |z - z_0||dz|.$$

by (3.3).

Now $z, z_0 \in S_k$, so

$$|z - z_0| \le \operatorname{diam} \tilde{S}_k = \frac{1}{2^k} \operatorname{diam} T.$$

Furthermore the length of S_k is

$$\operatorname{length}(S_k) = \frac{\operatorname{length}(T)}{2^k}.$$

Let $L = \operatorname{length}(T)$ and $D = \operatorname{diam} T$. Then

$$\begin{aligned}\left|\int_{S_k} f(z)dz\right| &\le \epsilon \int_{S_k} |z - z_0||dz| \\ &\le \epsilon \frac{D}{2^k} \int_{S_k} |dz| \\ &\le \epsilon \frac{D}{2^k}\frac{L}{2^k} = \epsilon \frac{D\,L}{4^k}.\end{aligned}$$

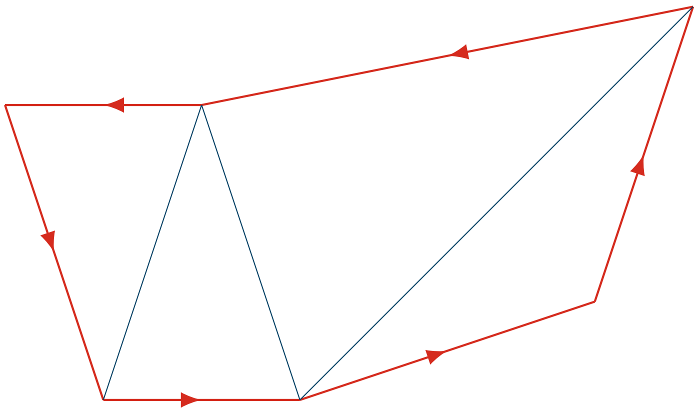


Figure 3.8: Cauchy's theorem for a polygonal curve

Combining this with (3.2) we find that for any $\epsilon > 0$ have

$$\left|\int_T f(z)dz\right| \leq 4^k \left|\int_{S_k} f(z)dz\right|$$
$$\leq 4^k \epsilon \frac{DL}{4^k}$$
$$= \epsilon DL.$$

Since $\epsilon > 0$ was arbitrary,

$$\int_T f(z)dz = 0$$

as required. □

Any polygon, such as the one drawn in Figure 3.8, can be decomposed into triangles – a process called triangulation. Integrating along a polygon can be written as the sum of a finite number of integrals along the triangles. As we have seen before sides of triangles appear twice in this decomposition but oriented in opposite directions, so their integrals along the common edges cancel. Therefore a consequence of Theorem 3.12 is the following.

Corollary 3.13. *Suppose that* f *is holomorphic in a simply connected domain* Ω*, and* C *is the perimeter of a polygon contained in* Ω*. Then*

$$\int_C f(z)dz = 0.$$

It remains to prove Cauchy's theorem in full generality.

The idea is to approximate the curve C by a sequence of polygons sufficiently well that the integrals along the polygons converge to the integral along C. The only thing we have to worry about is the convergence and so we need to be careful about how we define the polygon. Integral convergence is relatively weak and so we only need to employ the continuity of f, not the fact that it is holomorphic.

We will require the following result that strengthens the notion of continuity in metric spaces.

Lemma 3.14. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces and suppose that* $f\colon X \to Y$ *is a continuous function. Let* E *be a compact set in* (X, d_X) *then for all* $\epsilon > 0$ *there is a* $\delta > 0$ *such that for all* $x, y \in E$ *and* $d_X(x, y) < \delta$ *we have that* $d(f(x), f(y)) < \epsilon$*.*

The important thing here is that δ is independent of x and y as long as they are in the compact set E.

Proof. Suppose that $\epsilon > 0$. Since f is continuous we can find, for each $x \in E$ a $\delta(x)$ such that

$$f(B(x, \delta(x)) \subset B(f(x), \epsilon/2).$$

Now the family of open sets $\{B(x, \delta(x)/2)\colon x \in E\}$ is an open cover of E and so there is a finite sub cover

$$\{B(x_i, \delta(x_i)/2)\colon i = 1, 2, \dots, N\}.$$

Let $\delta = \min\{\delta(x_i)\colon i = 1, 2, \dots, N\}$.

Suppose that $x, y \in E$ are such that $d_X(x, y) < \delta$. Let i be chosen so that $x \in B(x_i, \delta(x_i)/2)$ then

$$\begin{aligned} d_X(y, x_i) &\le d_X(y, x) + d_X(x, x_i) \\ &< \frac{\delta}{2} + \frac{\delta(x_i)}{2} \\ &\le \frac{\delta(x_i)}{2} + \frac{\delta(x_i)}{2} = \delta(x_i). \end{aligned}$$

Therefore $y \in B(x_i, \delta(x_i))$ for some i. But then

$$\begin{aligned} d_Y(f(x), f(y)) &\le d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

The result is proved. □

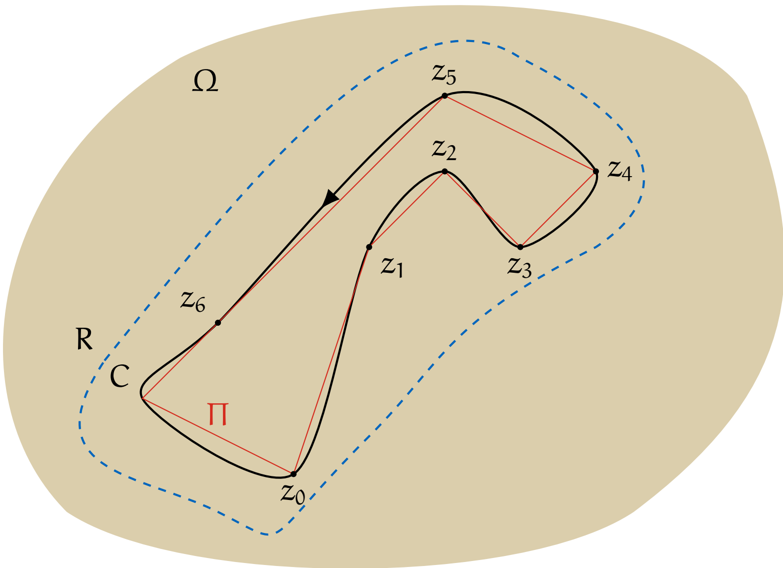


Figure 3.9: The proof of Cauchy's theorem

Proof of Cauchy's theorem. Since C is a simple closed curve in Ω we can find a compact set R such that $C \subset R \subset \Omega$. By Lemma 3.14 there is a $\delta > 0$ so that whenever $z, w \in R$,

$$|z - w| < \delta \qquad \Rightarrow \qquad |f(z) - f(w)| < \frac{\epsilon}{2L}$$

where $L = \text{length}(C)$.

Suppose that C is positively oriented and $P_\epsilon = \{z_0, z_1, \dots, z_n\}$ is a partition of C ordered so that z_{k+1} follows z_k on C for each $k = 0, 1, \dots, n-1$, P_ϵ is such that

$$\left| \int_C f(z)dz - \sum_{k=0}^{n} f(z_k)(z_{k+1} - z_k) \right| < \frac{\epsilon}{2}, \tag{3.4}$$

where we adopt the convention $z_{n+1} = z_0$ to avoid repetition.

We can choose P_ϵ so that the following hold, see Figure 3.9:

1. for each $k = 0, 1, \dots, n$ the line segment joining z_k to z_{k+1} is contained in R;
2. the polygon, Π, whose edges are the line segments z_k to z_{k+1}, $k = 0, 1, \dots, n$, does not intersect itself
3. for each k, $|z_{k+1} - z_k| < \delta$.

Only the second of these is controversial. It is possible of course that Π does intersect itself, but in this case we are left with a finite number of polygons, each of which satisfies the properties above. Since we only need the integral along Π to be 0 we can restrict the remainder of the proof to these finitely many polygons individually if needed.

Let Π_k be the part of Π consisting of the line segment from z_k to z_{k+1}, $k = 0, 1, \ldots, n$. We have that

$$\int_{\Pi} f(z)dz = \sum_{k=0}^{n} \int_{\Pi_k} f(z)dz.$$

We also have

$$\int_{\Pi_k} f(z_k)dz = f(z_k)(z_{k+1} - z_k).$$

Therefore

$$\begin{aligned}\left|\int_{\Pi} f(z)dz - \sum_{k=0}^{n} f(z_k)(z_{k+1} - z_k)\right| &= \left|\sum_{k=0}^{n} \int_{\Pi_k} (f(z) - f(z_k))dz\right| \\ &\leq \sum_{k=0}^{n} \int_{\Pi_k} |f(z) - f(z_k)|\,|dz|,\end{aligned}$$

by Proposition 3.6,

$$\begin{aligned}&\leq \frac{\epsilon}{2L} \sum_{k=0}^{n} \int_{\Pi_k} |dz| \\ &\leq \frac{\epsilon}{2L} L = \frac{\epsilon}{2}\end{aligned}$$

where we have used the fact that the length of Π is smaller than the length of C.

It follows by the triangle inequality that

$$\begin{aligned}\left|\int_C f(z)dz - \int_{\Pi} f(z)dz\right| &= \left|\int_C f(z)dz - \sum_{k=0}^{n} f(z_k)(z_{k+1} - z_k) + \sum_{k=0}^{n} f(z_k)(z_{k+1} - z_k) - \int_{\Pi} f(z)dz\right| \\ &\leq \left|\int_C f(z)dz - \sum_{k=0}^{n} f(z_k)(z_{k+1} - z_k)\right| + \left|\int_{\Pi} f(z)dz - \sum_{k=0}^{n} f(z_k)(z_{k+1} - z_k)\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

However by Corollary 3.13

$$\int_{\Pi} f(z)dz = 0$$

and so this reduces to

$$\left| \int_C f(z)dz \right| < \epsilon.$$

Since $\epsilon > 0$ the theorem is proved. □

Things to know

- How to find a parameterisation of standard examples of curves, such as a line segment or an arc of a circle;
- How to distinguish between the different types of curves as shown in Figure 3.3;
- How to integrate a given function along a curve;
- How to integrate a given function along a curve with respect to arc-length, especially the calculation of the length of a curve;
- The triangle inequality for integrals along a curve;
- The statement but not the proof of Cauchy's theorem.

Problems

1. To parameterise a line segment from a to b you can use

$$p(t) = a + t(b - a), \qquad 0 \leq t \leq 1.$$

Find a parameterisation for the following line segments.

   (a) The line from 1 to i
   (b) The line above in the opposite direction
   (c) The line from $1 + i$ to $-1 - i$
   (d) The line above in the opposite direction

2. To parameterise an arc of the boundary of $B(a, r)$ in the anti-clockwise direction

$$p(t) = a + re^{it}, \qquad \theta_1 \leq t \leq \theta_2.$$

Find the parameterisations of the following arcs in the anti-clockwise direction.

   (a) Centered at i from π to $3\pi/2$, radius 1;
   (b) Centred at $-i$ from 0 to π, radius 1.

3. To parameterise an arc of the boundary of $B(a, r)$ in the clockwise direction

$$p(t) = a + re^{i(\theta_1 + \theta_2 - t)}, \qquad \theta_1 \leq t \leq \theta_2.$$

Find the parameterisations of the following arcs in the clockwise direction.

   (a) Centered at i from π to $3\pi/2$;
   (b) Centred at $-i$ from 0 to π.

4. Find a parameterisation of the simple closed curve from -1 to 1 and then along a semi-circle in the upper half-plane (above the real axis) back to -1.

5. Sketch the following curves and find a parameterisation for each of them

   (a) The semi-circle of radius 3, anticlockwise
   (b) The semi-circle of radius 3, clockwise
   (c) The line segment from 0 to 2
   (d) The line segment from 0 to $1 + i$

6. Find the integral of $f(z) = z^2$ along the line segment joining $-i$ to i
7. Find the integral of $f(z) = z^3$ along the portion of the circle of radius 1 in the first quadrant, in the anti-clockwise direction
8. Find the integral of $f(z) = 1/z$ along the circle of radius 2 in the clockwise direction
9. Find the length of the curve parameterised by $p(t) = 3e^{2it} + 2$ for $-\pi \leq t \leq \pi$
10. Find the length of the curve parameterised by $p(t) = e^t \cos t + ie^t \sin t$ for $0 \leq t \leq 2\pi$
11. Find the length of the curve C parameterised by $p(t) = 10e^{it} - 2e^{5it}$, $0 \leq t \leq 2\pi$.
12. Suppose C_r is the circle of radius r taken in the anti-clockwise direction, and $|f(z)| \leq M$ for all $|z| \leq 1$. Use the triangle inequality to show

$$\lim_{r \to 0} \int_{C_r} f(z) dz = 0.$$

13. Verify Cauchy's theorem for the functions
    (a) $3z^2 + iz - 4$
    (b) $5 \sin 2z$

    if C is the square with vertices $1 \pm i$ and $-1 \pm i$.
14. If C is the boundary of the disk $B(2, 5)$, determine whether $\int_C \frac{1}{z-3} dz = 0$. Does your answer contradict Cauchy's theorem?
15. Given $g(z) = \int_{L(z)} \cos 3w \; dw$ where $L(z)$ is a curve joining $(\pi - \pi i)$ to z.
    (a) Prove that $g(z)$ is independent of the path joining $\pi - \pi i$ and z.
    (b) Determine $g(\pi i)$
    (c) Prove that $g'(z) = \cos 3z$

Chapter 4 : Applications of Cauchy's Theorem

"[The student] should understand mathematics as a living organism, a growing body of learning, fed by the efforts of many workers. If possible, he should acquire a sense of veneration for those who built up the structure that he is studying."

Analytic Function Theory, Vol I by Einar Hille.

4.1 Cauchy's integral formula

Cauchy's theorem has a number of important applications in complex analysis. In this chapter we will explore a few of them to extend what you have already learned about holomorphic functions.

We begin with the following remarkable result. First note that the Jordan curve theorem lets us assign an interior and exterior to a simple closed curve. A point ζ is said to be interior to a simple closed curve if it lies in its interior.

Theorem 4.1 – Cauchy's integral formula

Suppose that $f \in \mathcal{H}(\Omega)$ where Ω is a simply connected domain. Let C be a simple closed curve in Ω with ζ a point in its interior. Then

$$f(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \zeta} dz.$$

We will prove this when C is the boundary of a disk, although the full proof is not much more difficult. We begin with the following Lemma.

Lemma 4.2. *Suppose that* $f \in \mathcal{H}(\Omega)$ *and* $a \in \Omega$. *Then for* $\zeta \in B(a, r) \subset \Omega$ *we have*

$$\int_{\partial B(a,r)} \frac{1}{z - \zeta} dz = 2\pi i.$$

Proof. First we will caluclate the integral around the boundary $\partial B(\zeta, R)$ where $R < r - |z - \zeta|$ (this condition simply ensures that $B(\zeta, R) \subset B(a, r)$). We parameterise $\partial B(\zeta, R)$ as $p(t) = \zeta + re^{it}$

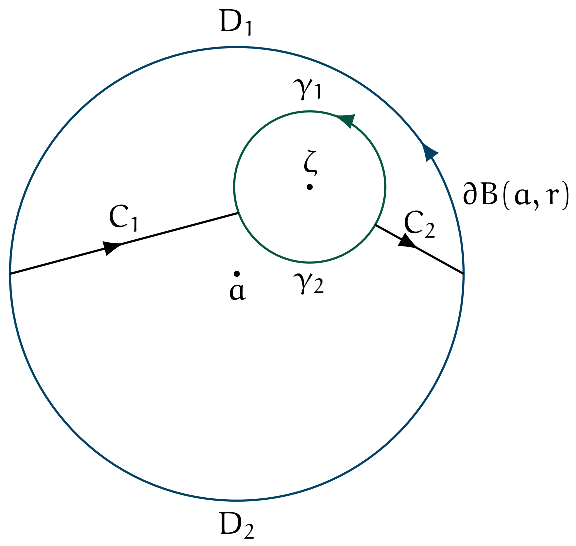


Figure 4.1: The proof of Lemma 4.2

for $0 \leq t < 2\pi$. Then

$$\begin{aligned}\int_{\partial B(\zeta,R)} \frac{1}{z-\zeta} dz &= \int_0^{2\pi} \frac{p'(t)}{p(t)-\zeta} dt \\ &= \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt \\ &= \int_0^{2\pi} i dt = 2\pi i.\end{aligned}$$

Suppose that $\zeta \neq a$ then we can draw two non-intersecting lines C_1 and C_2 joining $\partial B(a,r)$ and $\partial B(\zeta,R)$. Define γ_1 and γ_2 be the two parts of $\partial B(\zeta,R)$ between the points where C_1 and C_2 intersect it. Similarly let D_1 and D_2 be the two parts of $\partial B(a,r)$ between the points where C_1 and C_2 intersect this circle. See Figure 4.1.

Let E_1 be the path following C_1, then along $-\gamma_1$, then C_2 and finally D_1. Then E_1 is a simple closed curve and ζ is not in its interior. So $1/(z-\zeta)$ is holomorphic in the interior of E_1 and by Cauchy's theorem,

$$\int_{E_1} \frac{1}{z-\zeta} dz = 0.$$

In a similar way let E_2 be the curve comprising D_2 followed by $-C_2$ then $-\gamma_2$ and finally $-C_1$.

Again by Cauchy's theorem

$$\int_{E_2} \frac{1}{z-\zeta} dz = 0.$$

Therefore

$$0 = \int_{D_1} \frac{1}{z-\zeta} dz + \int_{C_1} \frac{1}{z-\zeta} dz - \int_{\gamma_1} \frac{1}{z-\zeta} dz + \int_{C_2} \frac{1}{z-\zeta} dz$$
$$0 = \int_{D_2} \frac{1}{z-\zeta} dz - \int_{C_1} \frac{1}{z-\zeta} dz - \int_{\gamma_2} \frac{1}{z-\zeta} dz - \int_{C_2} \frac{1}{z-\zeta} dz$$

If we now add these two equations then the integrals of C_1 and C_2 cancel and we get

$$\int_{D_1 \cup D_2} \frac{1}{z-\zeta} dz - \int_{\gamma_1 \cup \gamma_2} \frac{1}{z-\zeta} dz = 0.$$

Now $D_1 \cup D_2 = \partial B(a, r)$ and $\gamma_1 \cup \gamma_2 = \partial B(\zeta, R)$ and so from the previous calculation we see that

$$\int_{\partial B(a,r)} \frac{1}{z-\zeta} dz = \int_{\partial B(\zeta,R)} \frac{1}{z-\zeta} dz = 2\pi i$$

as required. □

Proof of Theorem 4.1. The proof uses the same trick that we used to prove the lemma. We will first show that

$$\int_{\partial B(a,r)} \frac{f(z) - f(\zeta)}{z-\zeta} dz = 0. \tag{4.1}$$

Using a similar argument as above we can show that

$$\int_{\partial B(a,r)} \frac{f(z) - f(\zeta)}{z-\zeta} dz = \int_{\partial B(\zeta,R)} \frac{f(z) - f(\zeta)}{z-\zeta} dz \tag{4.2}$$

for small enough R. As before we use the fact that $(f(z) - f(\zeta))/(z - \zeta)$ is holomorphic away from ζ, and so integrals of this function along simple closed curves are 0 by Cauchy's theorem.

Now f is continuous at ζ so given an $\epsilon > 0$ there is a $\delta > 0$ such that

$$|z - \zeta| < \delta \qquad \Rightarrow \qquad |f(z) - f(\zeta)| < \epsilon.$$

For $R < \delta$

$$\begin{aligned} \left| \int_{\partial B(\zeta,R)} \frac{f(z) - f(\zeta)}{z-\zeta} dz \right| &\le \int_{\partial B(\zeta,\delta)} \frac{|f(z) - f(\zeta)|}{|z-\zeta|} |dz| \\ &< \epsilon \int_{\partial B(\zeta,R)} \frac{1}{|z-\zeta|} |dz| \\ &= \epsilon \frac{1}{R} \int_{\partial B(\zeta,R)} |dz|, \end{aligned}$$

since $|z-\zeta| = R$ on $\partial B(\zeta, R)$,

$$= \epsilon\frac{1}{R}\,\mathrm{length}(\partial B(\zeta, R)) = \epsilon\frac{1}{R}2\pi R$$
$$= 2\pi\epsilon.$$

By (4.2) we have that

$$\left|\int_{\partial B(a,r)} \frac{f(z)-f(\zeta)}{z-\zeta}dz\right| = \left|\int_{\partial B(\zeta,R)} \frac{f(z)-f(\zeta)}{z-\zeta}dz\right| < 2\pi\epsilon$$

and since ϵ was arbitrary, (4.1) follows.

Now

$$\frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(z)}{z-\zeta}dz - f(\zeta) = \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(z)}{z-\zeta}dz - \frac{1}{2\pi i}f(\zeta)\int_{\partial B(a,r)} \frac{1}{z-\zeta}dz,$$

by Lemma 4.2,

$$= \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(z)}{z-\zeta}dz - \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(\zeta)}{z-\zeta}dz$$
$$= \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(z)-f(\zeta)}{z-\zeta}dz = 0,$$

and the result is proved. □

4.2 *Holomorphic functions are analytic*

When we defined analytic functions we noted that since you can differentiate power series, an analytic function is holomorphic. The converse is also true. A function that is holomorphic in a domain Ω is equal to its Taylor series in a neighbourhood of each point in Ω. In particular a holomorphic function is analytic, making the two definitions equivalent.

In this section we will prove this result. It is based on Cauchy's integral formula.

Theorem 4.3

Suppose that f is holomorphic in $B(z_0, r)$ for $z_0 \in \mathbb{C}$ and $r > 0$. Then $f(z)$ is equal to its Taylor series centred at z_0 for all $z \in B(z_0, r)$.

Lemma 4.4. *Let $z, z_0, h \in \mathbb{C}$ then*

$$\frac{1}{z-(z_0+h)} = \frac{1}{z-z_0} + \frac{h}{(z-z_0)^2} + \cdots + \frac{h^n}{(z-z_0)^{n+1}} + \frac{h^{n+1}}{(z-z_0)^{n+1}(z-(z_0+h))} \quad (4.3)$$

Proof. We begin with the geometric series formula:

$$(1-w)(1+w+\cdots+w^n) = 1-w^{n+1}.$$

If we substitute $w = h/(z-z_0)$ and rearrange, then

$$\left(1-\frac{h}{z-z_0}\right)\left(1+\frac{h}{z-z_0}+\cdots+\frac{h^n}{(z-z_0)^n}\right)+\frac{h^{n+1}}{(z-z_0)^{n+1}} = 1.$$

It follows that

$$\left(\frac{z-(z_0+h)}{z-z_0}\right)\left(1+\frac{h}{z-z_0}+\cdots+\frac{h^n}{(z-z_0)^n}\right)+\frac{h^{n+1}}{(z-z_0)^{n+1}} = 1$$

and so

$$(z-(z_0+h))\left(\frac{1}{z-z_0}+\frac{h}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^{n+1}}\right)+\frac{h^{n+1}}{(z-z_0)^{n+1}} = 1.$$

Dividing by $z-(z_0+h)$ then gives the result. □

Lemma 4.5. *Suppose that* $d>0$, $z \in \partial B(z_0,d)$ *and* $\zeta \in B(z_0,d)$ *then*

$$d-|\zeta-z_0| \le |z-\zeta| \le |\zeta-z_0|+d.$$

Proof. The diagram in Figure 4.2 demonstrates this. Thinking of z_0 and ζ as fixed points and z as moving around the circle, you should convince yourself that the distance from z to ζ is minimised and maximise precisely when z is at the opposite ends of the diameter passing through ζ. □

Proof of Theorem 4.3. We only need to show that f can be written as a power series with radius of convergence r. The fact that it is then its Taylor series follows from Corollary 2.14.

Let $d<r$. Then the closure of the ball $B(z_0,d)$ is contained in $B(z_0,r)$. We will show that $f(\zeta)$ can be written as a power series whenever $\zeta = z_0+h \in B(z_0,d)$.

First, by (4.3),

$$\frac{1}{z-\zeta} = \frac{1}{z-z_0}+\frac{h}{(z-z_0)^2}+\cdots+\frac{h^n}{(z-z_0)^{n+1}}+\frac{h^{n+1}}{(z-z_0)^{n+1}(z-\zeta)}.$$

Multiplying by $f(z)/(2\pi i)$ and integrating along the curve $C = \partial B(z_0,d)$ in the positive direction gives

$$\begin{aligned}
f(\zeta) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\zeta}dz &= \sum_{k=0}^{n}\left(\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{k+1}}dz\right)h^k + R_n(h) \\
&= \sum_{k=0}^{n}\left(\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{k+1}}dz\right)(\zeta-z_0)^k + R_n(h)
\end{aligned}$$

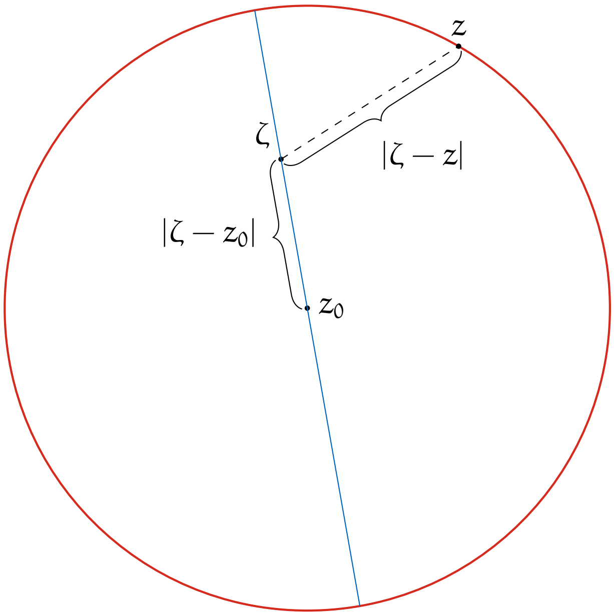


Figure 4.2: The proof of Lemma 4.5

where

$$R_n(h) = \frac{h^{n+1}}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}(z-\zeta)} dz.$$

Now

$$\begin{aligned} |R_n(h)| &\leq \frac{|h|^{n+1}}{2\pi} \int_C \frac{|f(z)|}{|z-z_0|^{n+1}|z-\zeta|} |dz| \\ &\leq \frac{|h|^{n+1}}{2\pi} \frac{M}{d^{n+1}} \frac{1}{d-|h|} \text{length}(C) \\ &= \frac{Md}{d-|h|} \left(\frac{|h|}{d} \right)^{n+1}, \end{aligned}$$

where $M = \sup_{z \in C} |f(z)|$ and we have used Lemma 4.5 to estimate $1/|z-\zeta| \leq 1/(d-|h|)$, and $\text{length}(C) = 2\pi d$.

It follows that $R_n(h) \to 0$ as $n \to \infty$ and the series converges to $f(\zeta)$:

$$f(\zeta) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{k+1}} dz \right) (\zeta - z_0)^k. \tag{4.4}$$

This is true for all $\zeta \in B(z_0, d)$ and so since $d < r$ was arbitrary this means $f(z)$ can be represented by its Taylor series, centred at z_0, for all $z \in B(z_0, r)$. □

A number of corollaries follow from this result.

Corollary 4.6. *Suppose that* $\Omega \subset \mathbb{C}$ *is a domain then* $\mathcal{H}(\Omega) = \mathcal{A}(\Omega)$.

Proof. Suppose that $z_0 \in \Omega$ is arbitrary. Then since Ω is open there is a $r > 0$ such that $B(z_0, r) \subset \Omega$. By the theorem $f(z)$ is equal to its Taylor series at all points in $B(z_0, r)$ and hence it is analytic at z_0. □

Combining this with Corollary 2.13 we get the following incredible result.

Corollary 4.7. *Suppose that* f *is complex-differentiable in* $B(z_0, r)$ *for some* z_0 *and* $r > 0$ *then* $f^{(k)}$ *exists at all points of* $B(z_0, r)$ *for all* k

In other words if a function is differentiable then it is infinitely differentiable!

If we now compare the description of the Taylor coefficients in Corollary 2.14 to (4.4) and rearrange we have the following consequence of the result, which is an extension of Cauchy's integral formula

Theorem 4.8 – Cauchy's integral formula for derivatives

Suppose that $f \in \mathcal{H}(\Omega)$ where Ω is a simply connected domain. Let C be a simple closed curve in Ω with ζ a point in its interior. Then

$$f^{(k)}(\zeta) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-\zeta)^{k+1}} dz.$$

And finally we put to bed the question of why the Taylor series of $1/(1+x^2)$ has radius of convergence 1 when it has continuous derivatives of all orders (see Example 10).

Corollary 4.9. *Suppose that the power series* $\sum_{n=0}^{\infty} a_n(z-a)^n$ *has radius of convergence* $R > 0$*. Then there is at least one point of* $\partial B(a, R)$ *at which* $f(z)$ *fails to be holomorphic.*

4.3 Zeros of holomorphic functions

A zero of a holomorphic function is a point z such that $f(z) = 0$. For example the function $z^2 + 1$ has zeros at $\pm i$. There are a couple of important results on the zeros of holomorphic functions that are worth mentioning here. They again highlight the difference between real analysis and complex analysis.

Theorem 4.10

Suppose $f \in \mathcal{H}(\Omega)$ and (z_m) is a sequence of zeros of f, i.e. for each m, $f(z_m) = 0$. If $z_m \to a$ as $m \to \infty$ and $a \in \Omega$ then $f(z) = 0$ for all $z \in \Omega$.

Proof. Since $a \in \Omega$, by Theorem 4.3 we may define $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, in $B(a, R)$, for $R > 0$. We will prove by induction that $a_n = 0$ for all n.

Since f is holomorphic it is continuous at a. Therefore $f(a) = f(\lim_{m\to\infty} z_m) = \lim_{m\to\infty} f(z_m) = 0$. So

$$0 = f(a) = a_0.$$

Now suppose then that $a_0, a_1, \ldots, a_k$ are all 0. Then

$$\begin{aligned} f(z) &= a_{k+1}(z-a)^{k+1} + a_{k+2}(z-a)^{k+2} + \cdots \\ &= (z-a)^{k+1}\left(a_{k+1} + a_{k+2}(z-a) + \cdots\right) \\ &= (z-a)^{k+1} g(z). \end{aligned}$$

Here $g(z) = a_{k+1} + a_{k+2}(z-a) + \cdots$ is a holomorphic function in $B(a, R)$ since it can be written as a power series. Note also that

$$g(z_m) = \frac{f(z_m)}{(z_m - a)^{k+1}} = 0$$

for all m and so by continuity $a_{k+1} = g(a) = \lim_{m\to\infty} g(z_m) = 0$.

It follows by induction that $a_n = 0$ for all n. □

Results about zeros always lead to results about uniqueness – i.e. results that give conditions under which two holomorphic functions are equal.

Corollary 4.11. *Suppose that* $f, g \in \mathcal{H}(\Omega)$. *If* $z_m \to a \in \Omega$, *as* $m \to \infty$, *is such that* $f(z_m) = g(z_m)$ *for all* m *then* $f = g$.

Another important consequence is the following

Corollary 4.12. *Suppose* $f \in \mathcal{H}(\Omega)$ *and* $f(a) = 0$. *Then there is a* $R > 0$ *such that* $f(z) \neq 0$ *for any* $z \in B(a, R)$ *except for* a.

Finally we have the following characterisation of zeros of holomorphic functions.

Corollary 4.13. *Suppose that* $f \in \mathcal{H}(\Omega)$ *and* $f(a) = 0$. *Then there is a* $k \geq 1$ *and* $R > 0$ *such that* $f(z) = (z-a)^k g(z)$ *where* $g \in \mathcal{H}(B(a, R))$ *and* $g(z) \neq 0$ *for all* $z \in B(a, R)$.

Proof. As in the proof of Theorem 4.10 if $f(a) = 0$ then we may write $f(z)$ as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

in a neighbourhood $B(a, R)$. By the previous corollary we also know that we can choose R small enough so that $f(z)$ has no other zeros in $B(a, R)$.

Since $f(a) = 0$ we have that $a_0 = 0$. If we let k be the smallest positive integer such that $a_k \neq 0$ then $a_0 = a_1 = \cdots = a_{k-1} = 0$ and

$$f(z) = \sum_{n=k}^{\infty} a_n (z-a)^n = (z-a)^k \sum_{n=k}^{\infty} a_n (z-a)^{n-k}.$$

The corollary now follows since $g(z) = \sum_{n=k}^{\infty} a_n (z-a)^{n-k}$ is holomorphic and non-zero in $B(a, R)$. □

Definition 4.14 – Multiplicity

The number k in the previous corollary is called the multiplicity of the zero a.

4.4 *Liouville's theorem*

A final consequence of Cauchy's theorem, or more precisely a consequence of Cauchy's integral formula, is an important estimate of entire functions that is surprisingly useful. It is a deep theorem of Liouville that is used broadly throughout mathematics.

Theorem 4.15 – Liouville's theorem

Suppose that f is an entire function and there is a $M > 0$ such that for all $z \in \mathbb{C}$,

$$|f(z)| \leq M.$$

Then $f(z)$ is constant throughout $\mathbb{C}$.

Proof. Let $k \geq 1$ and $\zeta \in \mathbb{C}$ be arbitrary. Let $C_R = \partial B(\zeta, R) = \{z \in \mathbb{C} \colon |z - \zeta| = R\}$. Then by

Theorem 4.8

$$
\begin{aligned}
|f^{(k)}(\zeta)| &= \left| \frac{k!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-\zeta)^{k+1}} dz \right| \\
&\leq \frac{k!}{2\pi} \int_{C_R} \frac{|f(z)|}{|z-\zeta|^{k+1}} |dz| \\
&\leq \frac{k!M}{2\pi} \frac{1}{R^{k+1}} \operatorname{length}(C_R) \\
&= k!M \frac{R}{R^{k+1}} = \frac{k!M}{R^k}, \qquad \text{since } \operatorname{length}(C_R) = 2\pi R.
\end{aligned}
$$

But this is arbitrarily small since $R^{-k} \to 0$ as $R \to \infty$. Therefore $f^{(k)}(\zeta) = 0$ for all $k \geq 1$. Since f is analytic, it is equal to its Taylor series in a neighbourhood of a given point ζ, but we have shown that the coefficients of the Taylor series are all 0 except possibly for the constant term. Therefore $f(z)$ is constant in a neighbourhood of each point and is therefore constant throughout $\mathbb{C}$. □

An immediate consequence of this is the fundamental theorem of algebra.

Theorem 4.16 – The fundamental theorem of algebra

Let $p(z) \in \mathbb{C}[z]$ have $\deg p \geq 1$. Then there is $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$ and p can be factorised as follows

$$p(z) = (z - \alpha)q(z)$$

where $q(z) \in \mathbb{C}[z]$.

The proof is straightforward. If $p(z) \neq 0$ then $1/p(z)$ is an entire function, we only then need to show it is bounded to deduce that it is constant and hence to produce a contradiction.

Proof. Writing $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ where $a_n \neq 0$ and $n > 0$ we have

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right) = z^n q(z).$$

Then $\lim_{z\to\infty} q(z) = a_n$ and so $|p(z)| \to \infty$ as $z \to \infty$. Thus there is a $R > 0$ such that $|p(z)| \geq 1$ for $z \in B(0, R)^*$, the exterior of $B(0, R)$.

Now let $f(z) = 1/p(z)$. By the argument above $|f(z)| \leq 1$ for $z \in B(0, R)^*$. Suppose that $p(z) \neq 0$ for any $z \in \mathbb{C}$, then $f(z)$ is holomorphic on the compact set $\overline{B(0, R)}$ and so is bounded, by $M > 1$ say. Then $|f(z)| \leq M$ for all $z \in \mathbb{C}$ and by Liouville's theorem it must therefore be constant. It follows that p is constant, which is a contradiction.

Therefore there is a $\alpha \in \mathbb{C}$ with $p(\alpha) = 0$ and the factorisation follows. □

Entire functions are different in many ways to functions that are holomorphic on a domain Ω strictly contained in $\mathbb{C}$. Complex analysis can be split between entire functions and non-entire functions. Two important theorems that illustrate this will be given here, they are unique to entire functions. For more information the classic book of Boas, [2], although difficult to find these days, is an excellent introduction.

Theorem 4.17. *A non-constant entire function comes arbitrarily close to every complex number.*

In other words given any $w \in \mathbb{C}$ and any $\epsilon > 0$ there is a z such that $|w - f(z)| < \epsilon$.

Proof. Suppose to the contrary that $|f(z) - w| \geq \epsilon$ for all $z \in \mathbb{C}$. Then $g(z) = 1/(f(z) - w)$ is entire and bounded by $1/\epsilon$. Therefore $g(z)$ is constant and, rearranging, it follows that $f(z)$ is constant, contradicting the hypothesis. □

Of course this doesn't say that an entire function takes every complex value, it just says it comes arbitrarily close to them. However we state here the following fundamental theorem that says that there can only be one point that an entire function doesn't take.

Theorem 4.18 – Picard's theorem

A non-constant entire function assumes each complex value, with one possible exception.

Things to know

- Cauchy's integral formula, both the standard one and the one for derivatives, and how to apply them to find a given integral;
- The notion that Holomorphic functions and Analytic functions are the same, and the consequences of that;
- The properties of the zeros of a holomorphic function, in particular from Theorem 4.10, Corollary 4.12 and Corollary 4.13;
- The definition of multiplicity;
- The statement and proof of Liouville's theorem;
- Applications of Liouville's theorem to uniqueness problems in complex analysis.

Problems

1. State the following integrals (no calculation is needed). In each case C is the boundary of the unit disk $B(0, 1)$.

   (a) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{1}{z}\,dz$;

   (b) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{1}{z^2}\,dz$;

   (c) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{z^3}{z-1/2}\,dz$;

   (d) $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{z^3}{(z-1/2)^2}\,dz$.

2. Suppose C is the closed curve parameterised by $p(t) = e^{it}$ for $0 \leq t \leq 4\pi$. Show that

$$\frac{1}{2\pi i}\int_C \frac{1}{z}\,dz = 2.$$

   Why isn't this a contradiction of Cauchy's integral formula?

3. Verify Cauchy's integral formula for the function $f(z) = z^n$ in the disk $B(0, 1)$. That is show that

$$\zeta^n = \frac{1}{2\pi i}\int_C \frac{z^n}{z-\zeta}\,dz.$$

   Hint: write $z^n = z^n - \zeta^n + \zeta^n$, split the integral and use Cauchy's theorem and Lemma 4.2.

4. Prove that if $f(z)$ is holomorphic in $B(0, R)$ for some $R > 1$, then the coefficients a_n of its power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta.$$

5. Suppose that $f(z)$ is holomorphic in $B(0, 1)$ and its set of zeros is closed, prove that $f(z) \equiv 0$.

6. Suppose $f(z)$ is entire and $|f(z)| \leq |\exp z| = e^{\operatorname{Re} z}$ for all $z \in \mathbb{C}$. Show that there is a λ with $|\lambda| = 1$ such that $f(z) = \lambda \exp z$.

7. Extend Liouville's theorem to the following. Suppose that $|f(z)| \leq M|z|^n$ for all $z \in \mathbb{C}$ and some $M > 0$. Then $f(z)$ is a polynomial and $\deg f(z) \leq n$.

Chapter 5 : The Residue Theorem

"The shortest route between two truths in the real domain passes through the complex domain."

Jacques Salomon Hadamard (1865-1963)

5.1 *Singularities*

In much of what we've done so far we have been studying holomorphic functions in a domain Ω. Applications of Cauchy's theorem require functions to be holomorphic.

In this chapter we're going to extend some of the ideas to functions that are not holomorphic everywhere but have isolated points where they are not holomorphic. We've already seen an example of this when we saw Cauchy's integral formula:

$$f^{(n)}(\zeta) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-\zeta)^{n+1}} dz, \qquad n \geq 0.$$

The integrand here, $\dfrac{f(z)}{(z-\zeta)^{n+1}}$ is holomorphic in a domain containing C *except* at the point ζ.

Example 15. Let $f(z) = 1/z^2$ defined in $B(0, 1)$.

In the unit disk $B(0, 1)$ the function is defined and holomorphic everywhere except at the point 0. At 0 the denominator has a zero of multiplicity 2.

It turns out that this example perfectly illustrates the possibilities when we have a function $f(z)$ that is defined everywhere except possibly at *isolated* singularities. You may notice a similarity between these *singularities* and zeros. In the same way as we used power series to characterise the multiplicity of zeros we will use an extension of the Taylor series to define and characterise singularities.

Definition 5.1 – Laurent series

If $f(z)$ is holomorphic in the annulus $\{z \in \mathbb{C} \colon r < |z-a| < R\}$ then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \qquad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\zeta)^{n+1}} dz$$

and $C = \{z \in \mathbb{C} \colon |z-a| = \rho$ for some $r < \rho < R$.

The series is called a Laurent series, it is seemingly the same as a Taylor series except that the terms include negative indexes as well. In practise there are other ways of finding the series rather than using the integral.

Example 16. Let $f(z) = \frac{1}{z^2} \exp z$.

Here

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) \\ &= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} \\ &= \sum_{n=-2}^{\infty} \frac{1}{(n+2)!} z^n. \end{aligned}$$

Note here we have that $f(z)$ is analytic everywhere in the complex plane except the point 0. So our annulus here is

$$\{z \in \mathbb{C} : 0 < |z| < \infty\}.$$

The point 0 is an example of an *isolated singularity*.

Definition 5.2 – Isolated singularities

Suppose that $f(z)$ is defined by its Laurent series in the annulus $\{z \in \mathbb{C} : 0 < |z - a| < R\}$ for $R \leq \infty$. Then a is an *isolated singularity* for $f(z)$.

The nature of the Laurent series determines the behaviour of the singularity. There are three distinct possibilities here.

1. *No negative powers.*

   In this case we have that $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$. If we merely define $f(a) = a_0$ here then we can extend our function from the annulus to the disk $B(a, R)$. Furthermore the function is now holomorphic since it can be defined in terms of a power series. So in this case the point a is not a problem at all and can be dealt with quite easily, we call these kinds of points *removable singularities*.

2. *A finite number of negative powers.*

   This time we have $a_n = 0$ for all $n < -m$ but $a_{-m} \neq 0$, for some $m > 0$. So we may write the Laurent series as

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \cdots.$$

Here our function tends to infinity as $z \to a$ and so it cannot be dealt with as we did with the last case. We say that $f(z)$ has a *pole of order* m at a.

Since it is not always possible or efficient to write a function as its Laurent series we prefer a different definition of a pole. Note that if $f(z)$ has a pole of order m then

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + a_{-m+2}(z-a)^2 + \cdots .$$

The following definition of a pole is, then, equivalent.

Definition 5.3. Suppose that $f(z)$ is holomorphic in $\{z \in \mathbb{C} \colon 0 < |z-a| < R\}$ for some $R > 0$. Then $f(z)$ is said to have a pole of order m if $(z-a)^m f(z)$ has a removable singularity at a and m is the smallest such integer.

Example 17. Let $f(z) = \dfrac{\sin z}{z}$ in $\{z \in \mathbb{C} \colon 0 < |z| < \infty\}$.

This time

$$\begin{aligned} f(z) &= \frac{1}{z}\left(\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}\right) \\ &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots . \end{aligned}$$

This function has a removable singularity at 0.

Example 18. Let $f(z) = \dfrac{\cos z}{z^2}$ in $\{z \in \mathbb{C} \colon 0 < |z| < \infty\}$.

Here

$$\begin{aligned} f(z) &= \frac{1}{z^2}\left(\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}\right) \\ &= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} + \cdots . \end{aligned}$$

This function has a pole of order 2 at 0.

Note for this function we could also note that

$$zf(z) \to \infty, \qquad z \to 0$$

but $z^2 f(z) = \cos z$ has a removable singularity at 0. So $f(z)$ has a pole of order 2 at 0.

Example 19. Let $f(z) = \dfrac{1}{1+z^2}$ for $\{z \in \mathbb{C}\colon 0 < |z - i| < 2\}$.

Here it is less easy to define $f(z)$ as a Laurent series. Instead note that

$$(z-i)f(z) = (z-i)\frac{1}{(z+i)(z-i)} = (z+i) \to 2i, \qquad z \to i$$

and so $f(z)$ has a pole of order 1 at i.

3. *Infinitely many negative terms.*

   This case is the worst in terms of behaviour of the function. We say that $f(z)$ has an *essential singularity* at a.

   Example 20. Let $f(z) = \exp(1/z)$ for $\{z \in \mathbb{C}\colon 0 < |z| < \infty\}$.

   This time

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}. \end{aligned}$$

   The function $f(z)$ has an essential singularity at 0.

5.2 The Residue Theorem

We would like to extend Cauchy's theorem to functions that may have isolated singularities in the interior of a simple closed curve C. We can't expect to have an integral equating to 0 here but we might expect something that doesn't depend on C. We will begin with the following lemma.

Lemma 5.4. *Suppose that C is a simple closed curve, positively oriented, and a is a point in the interior of C. Then*

$$\int_C \frac{1}{(z-a)^n}\,dz = \begin{cases} 2\pi i, & n = 1; \\ 0, & n \neq 1. \end{cases}$$

Proof. As in the proof of Lemma 4.2 we first show that we may replace C with the boundary of a disk centred at a. Let $R > 0$ be chosen so that $B(a, R)$ is contained in the interior of C. Join the boundary of $B(a, R)$ to two arbitrary points on C by lines γ_1 and γ_2 as shown in Figure 5.1. Now

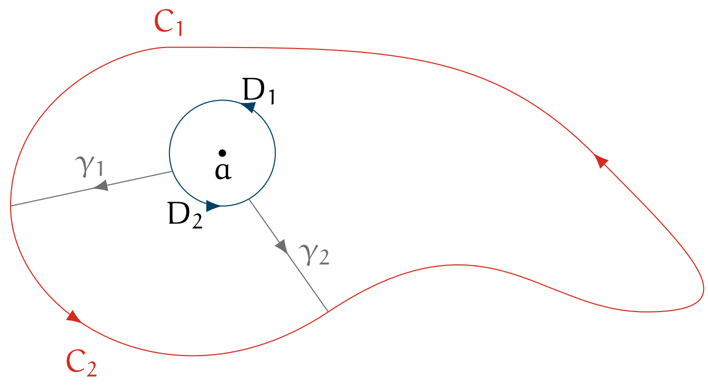


Figure 5.1: Proof of Lemma 5.4.

split the curve C into C_1 and C_2 at the points where γ_1 and γ_2 intersect C, and the boundary of $B(a, R)$ into D_1 and D_2.

Now let $E = C_1 \cup (-\gamma_1) \cup (-D_1) \cup \gamma_2$ then a is not in the interior of this simple closed curve so

$$\int_E \frac{1}{(z-a)^n} dz = 0$$

by Cauchy's theorem. The same is true along the simple closed curve $F = C_2 \cup (-\gamma_2) \cup (-D_2) \cup \gamma_1$. Therefore

$$0 = \int_{C_1} \frac{1}{(z-a)^n} dz - \int_{\gamma_1} \frac{1}{(z-a)^n} dz - \int_{D_1} \frac{1}{(z-a)^n} dz + \int_{\gamma_2} \frac{1}{(z-a)^n} dz$$
$$0 = \int_{C_2} \frac{1}{(z-a)^n} dz + \int_{\gamma_1} \frac{1}{(z-a)^n} dz - \int_{D_2} \frac{1}{(z-a)^n} dz - \int_{\gamma_2} \frac{1}{(z-a)^n} dz.$$

Adding these together and noting that $C = C_1 \cup C_2$ and $\partial B(a, R) = D_1 \cup D_2$ we find that

$$\int_C \frac{1}{(z-a)^n} dz - \int_{\partial B(a,R)} \frac{1}{(z-a)^n} dz = 0$$

from which we get

$$\int_C \frac{1}{(z-a)^n} dz = \int_{\partial B(a,R)} \frac{1}{(z-a)^n} dz.$$

Now if we parameterise $\partial B(a, R)$ by $p(t) = a + Re^{it}$ for $0 \leq t \leq 2\pi$ then $p'(t) = Rie^{it}$ and

$$\begin{aligned}\int_{\partial B(a,R)} \frac{1}{(z-a)^n} dz &= \int_0^{2\pi} \frac{1}{Re^{int}} Rie^{it} dt \\ &= \int_0^{2\pi} ie^{i(1-n)t} dt.\end{aligned}$$

If $n = 1$ then the integrand reduces to i and so the integral is $2\pi i$. If $n \neq 1$ then we get

$$\int_0^{2\pi} ie^{i(1-n)t} dt = \left[\frac{1}{1-n} e^{i(1-n)t}\right]_0^{2\pi} = \frac{1}{1-n}(1 - e^{2(1-n)\pi i}) = 0.$$

□

Consider a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

Let C be a simple closed curve, positively oriented, with a in its interior (but no other singularities). The assuming that we can swap the summation and the integral around we should get

$$\begin{aligned}\int_C f(z) dz &= \int_C \left(\sum_{n=-\infty}^{\infty} a_n (z-a)^n \right) dz \\ &= \sum_{n=-\infty}^{\infty} \left(\int_c a_n (z-a)^n dz \right) \\ &= 2\pi i a_{-1}\end{aligned}$$

by Lemma 5.4.

The value a_{-1} is called the *Residue* of $f(z)$ at a. This equation gives us a method for evaluating the integral of a function with a pole or essential singularity.

Example 21. Let C be simple closed curve, positively oriented, with 0 in its interior. Find the integral along C of the function $f(z) = \exp(1/z)$.

We have previously calculated

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

and so here the residue at 0 is the coefficient of z^{-1} which is 1. Therefore

$$\int_C f(z) dz = 2\pi i \times 1 = 2\pi i.$$

Example 22. Let C be simple closed curve, positively oriented, with 0 in its interior. Find the integral along C of the function $f(z) = \dfrac{\cos z}{z^2}$.

We previously found that

$$f(z) = \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} + \cdots.$$

and so this time $a_{-1} = 0$ and so

$$\int_C f(z)dz = 2\pi i \times 0 = 0.$$

A better method for finding residues is needed in most practical situations. Suppose that $f(z)$ has a pole of order m at a then

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-a)^n$$

and a_{-1} is the residue at a. Now

$$(z-a)^m f(z) = \sum_{k=0}^{\infty} a_{n-m}(z-a)^n = g(z)$$

and then a_{-1} is the $(m-1)$th coefficient of the power series for $g(z)$. So

$$a_{-1} = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

This is the basis for the following definition when a is a pole.

Definition 5.5 – Residues for poles

Suppose that $f(z)$ has a pole of order m at a. Then the residue at a is

$$\operatorname{Res}_a f(z) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z).$$

We will illustrate the use of this formula with the following example.

Example 23. Let $f(z) = (z^4 - z^3 - 17z + 2)/(z-1)^3$.

Here we have a pole at 1 of order 3 (be careful here! You should check that $(z-1)$ is not a factor of the numerator).

$$\begin{aligned}\operatorname{Res}_1 f(z) &= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}(z-1)^3 f(z)\\ &= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}(z^4-z^3-17z+2)\\ &= \frac{1}{2}\lim_{z\to 1}(12z^2-6z)\\ &= \frac{1}{2}\times 6 = 3.\end{aligned}$$

Therefore if C is any simple closed curve, positively oriented, and containing 1 in its interior, then

$$\int_C f(z)dz = 2\pi i \times 3 = 6\pi i.$$

Some functions have many poles of different orders. We can extend the method above to this scenario.

Theorem 5.6 – Cauchy's Residue Theorem

Suppose $f(z)$ is holomorphic in a domain Ω except for poles. If C is a simple closed curve with poles $z_1, z_2, \ldots, z_n$ in the interior of C then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f(z).$$

Proof. We will only sketch the proof. Let r_k be chosen small enough so that the disks $B(z_k, r_k)$ do not overlap and are small enough to be in the interior of C. See Figure 5.2. In the same way as we did in the proofs of Lemmas 4.2 and 5.4 we can replace the integral along C with the sum of the integrals along the boundaries $\partial B(z_k, r_k)$.

Then

$$\int_{\partial B(z_k, r_k)} f(z)dz = 2\pi i \operatorname{Res}_{z_k} f(z)$$

and the theorem is proved. □

5.3 Simple poles

A pole of order 1 is called a *simple pole*. For simple poles the calculation of the residue is more simple. Here we have

$$\operatorname{Res}_a f(z) = \lim_{z\to a}(z-a)f(z).$$

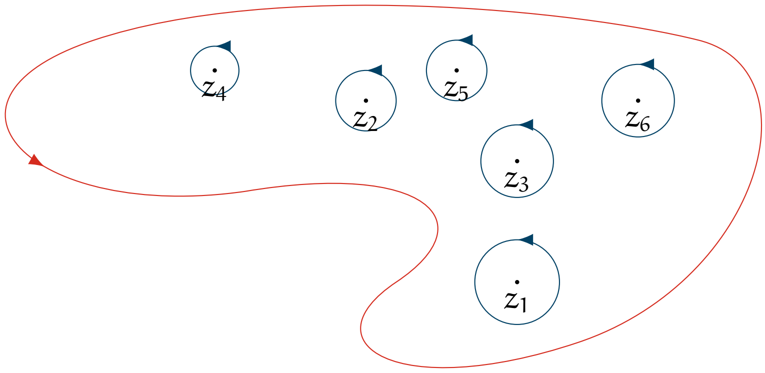


Figure 5.2: Proof of Cauchy's Residue Theorem

Example 24. Find the residue at $2i$ of $f(z) = (z^2 - 4)/(z^2 + 4)$.

This is a simple pole so

$$\begin{aligned}
\operatorname{Res}_{2i} f(z) &= \lim_{z \to 2i} (z - 2i) \frac{(z^2 - 4)}{(z^2 + 4)} \\
&= \lim_{z \to 2i} (z - 2i) \frac{(z^2 - 4)}{(z + 2i)(z - 2i)} \\
&= \lim_{z \to 2i} \frac{(z^2 - 4)}{(z + 2i)} \\
&= \frac{(2i)^2 - 4}{2i + 2i} \\
&= \frac{-8}{4i} = 2i.
\end{aligned}$$

Example 25. Find the residues of $f(z) = 1/(1 + z^3)$ at each of its poles.

The poles are the cube roots of -1 which are

$$\omega_k = e^{i(2\pi k + \pi/2)/3}, \qquad k = 0, 1, 2.$$

We will factorise $(1 + z^3) = (z - \omega_0)(z - \omega_1)(z - \omega_2)$ then

$$\operatorname{Res}_{\omega_k} f(z) = \lim_{z \to \omega_k} (z - \omega_k) \prod_{i=0}^{2} \frac{1}{z - \omega_i} = \prod_{\substack{i=1 \\ i \neq k}}^{2} \frac{1}{\omega_k - \omega_i}.$$

An useful tool for calculating residues, and limits of ratios in general, is l'Hopital's rule. This is a standard tool in calculus but has a particularly nice statement in complex analysis.

Theorem 5.7. *Suppose that* $f(z)$ *and* $g(z)$ *are holomorphic in a disk* $B(a, r)$ *and* a *is a zero of multiplicity* m *for both* $f(z)$ *and* $g(z)$. *Then*

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{(m)}(a)}{g^{(m)}(a)}.$$

Proof. By Corollary 4.13 we may write

$$f(z) = (z-a)^m \sum_{n=0}^{\infty} a_{n+m}(z-a)^n, \qquad g(z) = (z-a)^m \sum_{n=0}^{\infty} b_{n+m}(z-a)^n.$$

By Corollary 2.14,

$$a_k = \frac{f^{(k)}(a)}{k!}, \qquad b_k = \frac{g^{(k)}(a)}{k!}, \qquad k = m, m+1, \ldots.$$

Therefore

$$\begin{aligned}
\lim_{z \to a} \frac{f(z)}{g(z)} &= \lim_{z \to a} \frac{(z-a)^m \sum_{n=0}^{\infty} a_{n+m}(z-a)^n}{(z-a)^m \sum_{n=0}^{\infty} b_{n+m}(z-a)^n} \\
&= \lim_{z \to a} \frac{\sum_{n=0}^{\infty} a_{n+m}(z-a)^n}{\sum_{n=0}^{\infty} b_{n+m}(z-a)^n} \\
&= \frac{a_m}{b_m} = \frac{f^{(m)}(a)/m!}{g^{(m)}(a)/m!} \\
&= \frac{f^{(m)}(a)}{g^{(m)}(a)}.
\end{aligned}$$

□

Example 26. Consider again the residues of $f(z) = 1/(1+z^3)$ at each of its poles.

We'll use l'Hopital's rule this time. First

$$\begin{aligned}
\operatorname{Res}_{\omega_k} f(z) &= \lim_{z \to \omega_k} (z - \omega_k) f(z) \\
&= \lim_{z \to \omega_k} \frac{z - \omega_k}{1 + z^3} \\
&= \frac{1}{3\omega_k^2}.
\end{aligned}$$

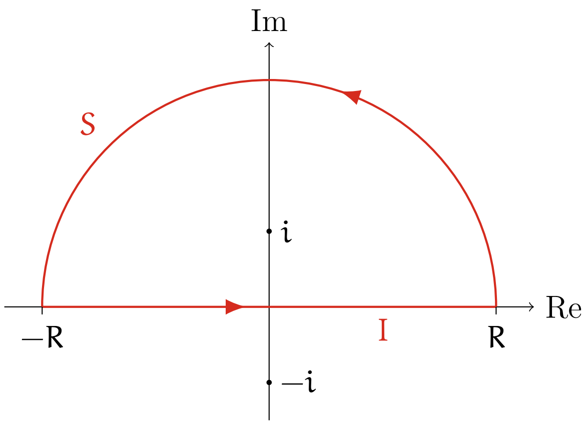


Figure 5.3: The closed contour in Example 27

We can simplify this even further by noticing that $-1 = \omega_k^3 = \omega_k^2\omega_k$ so that

$$\mathrm{Res}_{\omega_k} f(z) = -\frac{1}{3}\omega_k.$$

Aside: the previous formulation of the residue in this example leads to the following interesting formula

$$\prod_{\substack{i=1 \\ i\neq k}}^{2} (\omega_k - \omega_i) = 3\omega_k^2.$$

5.4 Applications of the Residue Theorem to real integrals

One of the most commonly seen applications of the residue theorem is in the evaluation of previously difficult real integrals. The best way to learn about this is to see an example.

Example 27. Find the value of the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

We will integrate $f(z) = 1/(1+z^2)$ along the curve consisting of the semi-circle of radius $R > 0$, centred at 0 in the upper half-plane, and the line-segment from $-R$ to R, see Figure 5.3. Call this curve C. As long as $R > 1$ we can see from the diagram that C contains only the simple pole at i. Therefore by the residue theorem

$$\begin{aligned}\int_C \frac{1}{1+z^2}\,dz &= 2\pi i \operatorname{Res}_i f(z)\\ &= 2\pi i \lim_{z\to i}\frac{z-i}{1+z^2}\\ &= 2\pi i \lim_{z\to i}\frac{1}{2z} = 2\pi i\frac{1}{2i}\\ &= \pi.\end{aligned}$$

Now this integral does not depend on R so we can choose R to be anything at all. In particular we could let $R \to \infty$. The reason we might want to do this is that the integral along the curve C can be split between the integral along the line segment I from $-R$ to R, and the semi-circle S. Then letting $R \to \infty$

$$\int_I \frac{1}{1+z^2}\,dz = \int_{-R}^{R}\frac{1}{1+x^2}\,dx \to \int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx.$$

Here the integral converges by Lebesgue's dominated convergence theorem: Define

$$f_R(x) = \mathbb{1}_{[-R,R]}\frac{1}{(1+x^2)}.$$

Then $f_R(x) \leq f(x)$ for all $x \in \mathbb{R}$ and $1/(1+x^2)$ is integrable on $\mathbb{R}$.

We will show that

$$\lim_{R\to\infty}\int_S \frac{1}{1+z^2}\,dz = 0.$$

As we have done a number of times we will do this using the triangle inequality.

$$\begin{aligned}\left|\int_S \frac{1}{1+z^2}\,dz\right| &\leq \int_S \frac{1}{|1+z^2|}|dz|\\ &\leq \int_S \frac{1}{|z|^2-1}|dz|,\end{aligned}$$

since $|z^2+1| \geq |z|^2-1$ by the triangle inequality,

$$\begin{aligned}&= \frac{1}{R^2-1}\int_S |dz| = \frac{1}{R^2-1}\,\text{length}(S)\\ &= \frac{\pi R}{R^2-1}.\end{aligned}$$

Therefore since the last expression tends to 0 as $R \to \infty$ we have that

$$\lim_{R\to\infty}\int_S \frac{1}{1+z^2}\,dz = 0$$

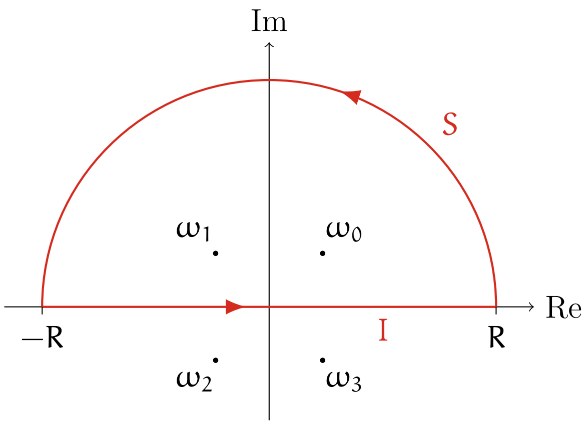


Figure 5.4: The curve in Example 28.

as required.

Collecting this together we have that for all $R > 1$

$$\pi = \int_C \frac{1}{1+z^2} = \int_{-R}^{R} \frac{1}{1+x^2} dx + \int_S \frac{1}{1+z^2} dz.$$

If we now take the limit as $R \to \infty$ we get

$$\pi = \lim_{R\to\infty} \pi = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{1+x^2} dx + \int_S \frac{1}{1+z^2} dz \right) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

So

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

Example 28. Using the residue theorem find

$$\int_0^{\infty} \frac{x^2}{1+x^4} dx.$$

This time we are integrating from 0 to infinity, however notice that $x^2/(1+x^4)$ is an even function and so

$$\int_0^{\infty} \frac{x^2}{1+x^4} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx.$$

We can therefore find this integral by considering a curve like the one we had in the last example, see Figure 5.4.

The poles of $z^2/(1+z^4)$ are the fourth roots of -1 which are

$$\omega_k = e^{i(2\pi k+\pi)/4}, \qquad k = 0, 1, 2, 3,$$

also drawn in Figure 5.4. Here only

$$\omega_0 = \frac{1}{\sqrt{2}}(1+i), \qquad \omega_1 = \frac{1}{\sqrt{2}}(-1+i)$$

are contained in the interior of the curve C. Therefore by the residue theorem

$$\int_C \frac{z^2}{1+z^4}\,dz = 2\pi i\left(\mathrm{Res}_{\omega_0} f(z) + \mathrm{Res}_{\omega_1} f(z)\right).$$

Both these are simple poles so we can use l'Hopital's rule to find them.

$$\begin{aligned}
\mathrm{Res}_{\omega_0} f(z) &= \lim_{z\to\omega_0}(z-\omega_0)f(z) \\
&= \lim_{z\to\omega_0}\frac{z^2(z-\omega_0)}{1+z^4} \\
&= \frac{3\omega_0^2-\omega_0(2\omega_0)}{4\omega_0^3} \\
&= \frac{\omega_0^2}{4\omega_0^3} \\
&= \frac{1}{4\omega_0} = \frac{\sqrt{2}}{4(1+i)} \\
&= \frac{\sqrt{2}}{8}(1-i).
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{Res}_{\omega_1} f(z) &= \lim_{z\to\omega_1}(z-\omega_1)f(z) \\
&= \frac{3\omega_1^2-\omega_1(2\omega_1)}{4\omega_1^3} \\
&= \frac{1}{4\omega_1} = \frac{\sqrt{2}}{4(-1+i)} \\
&= \frac{\sqrt{2}}{8}(-1-i).
\end{aligned}$$

Therefore

$$\mathrm{Res}_{\omega_0} f(z) + \mathrm{Res}_{\omega_1} f(z) = \frac{\sqrt{2}}{8}(1-i) + \frac{\sqrt{2}}{8}(-1-i) = -i\frac{\sqrt{2}}{4}.$$

It follows that

$$\int_C \frac{z^2}{1+z^4}\,dz = 2\pi i\left(-i\frac{\sqrt{2}}{4}\right) = \pi\frac{\sqrt{2}}{2}.$$

As before we now have to show that

$$\lim_{R\to\infty}\int_C \frac{z^2}{1+z^4}\,dz = \int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx.$$

First if I is the line segment joining $-R$ to R then

$$\lim_{R\to\infty}\int_I f(z)\,dz = \int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx.$$

Second let S be the semi-circular part of C. Then

$$\begin{aligned}\left|\int_S f(z)\,dz\right| &\le \int_S \left|\frac{z^2}{1+z^4}\right||dz| \\ &\le \int_S \frac{|z|^2}{|z|^4-1}|dz| \\ &= \frac{R^2}{R^4-1}\pi R \\ &= \frac{\pi R^3}{R^4-1}.\end{aligned}$$

And it follows that

$$\lim_{R\to\infty}\int_S f(z)\,dz = 0.$$

So as before we have

$$\frac{\sqrt{2}}{2}\pi = \lim_{R\to\infty}\int_C \frac{z^2}{1+z^4}\,dz = \int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx.$$

Finally we have

$$\int_0^{\infty}\frac{x^2}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx = \frac{\sqrt{2}}{4}\pi.$$

This method can be used for finding real integrals of the form

$$\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\,dx$$

when p and q are polynomials with $\deg p \le \deg q - 2$ (i.e. the degree of p is at least two less than the degree of q) as long as this integral exists and $q(x) \neq 0$ for any $x \in \mathbb{R}$. The strategy is to integrate over a closed curve like the one in Figure 5.3, calculate the residues and show that the integral along the semi-circle tends to 0 as $R \to \infty$.

Residues can also be used to calculate other types of real integrals. In general the choice of closed curve needs to be chosen wisely so that the 'complex part' of the integral tends to 0. There are a number of general methods for doing this.

5.4.1 Integrals of the form $\int_0^{2\pi} G(\cos t, \sin t)\,dt$

We will use the standard relationships

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \qquad \sin t = \frac{e^{it} - e^{-it}}{2i}$$

to rewrite integrals of this form as an integral over a circular path. We already know that we can parameterise these integrals using the parameterisation

$$p(t) = ae^{it}, \qquad 0 \le t \le 2\pi,$$

so our aim is to rewrite the integral in the form

$$\int_0^{2\pi} f(p(t))p'(t)\,dt = \int_C f(z)\,dz.$$

We will demonstrate the technique with the following example.

Example 29. Evaluate the integral

$$\int_0^{2\pi} \frac{1}{2 + \sin t}\,dt.$$

Let $p(t) = e^{it}$ then we have

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{p(t) + 1/p(t)}{2},$$

and

$$\sin t = \frac{e^{it} - e^{-it}}{2i} = \frac{p(t) - 1/p(t)}{2i}.$$

Here we have used the fact that $e^{-it} = 1/e^{it} = 1/p(t)$.

The denominator in the integrand is

$$\begin{aligned}2+\sin t &= 2+\frac{p(t)-1/p(t)}{2i}\\ &= \frac{1}{p(t)}\left(2p(t)+\frac{1}{2i}(p(t)^2-1)\right).\end{aligned}$$

Now $p'(t) = ie^{it} = ip(t)$ so that $p(t) = p'(t)/i$ and so we can rewrite the integrand above as

$$\begin{aligned}\frac{1}{2+\sin t} &= \frac{p(t)}{\left(2p(t)+\frac{1}{2i}(p(t)^2-1)\right)}\\ &= \frac{p'(t)}{i\left(2p(t)+\frac{1}{2i}(p(t)^2-1)\right)}\\ &= \frac{2p'(t)}{p(t)^2+4ip(t)-1}.\end{aligned}$$

The reason we rearrange the integral as we have done is so that we may now write this as

$$\int_0^{2\pi}\frac{2p'(t)}{p(t)^2+4ip(t)-1}\,dt = \int_C \frac{2}{z^2+4iz-1}\,dz$$

where C is the boundary of the unit disk $B(0,1)$. We can use the quadratic formula (which holds for complex quadratics as well) to find that the poles are where the denominator has zeros. They are simple poles at

$$a = (-2+\sqrt{3})i, \qquad b = (-2-\sqrt{3})i.$$

Only the first is in $B(0,1)$ and its residue is

$$\begin{aligned}\operatorname{Res}_a \frac{2}{z^2+4iz-1} &= \operatorname{Res}_a \frac{2}{(z-a)(z-b)}\\ &= \frac{2}{a-b} = \frac{2}{(-2+\sqrt{3})i-(-2-\sqrt{3})i}\\ &= \frac{1}{\sqrt{3}i}.\end{aligned}$$

It follows by the Residue Theorem that

$$\int_0^{2\pi}\frac{1}{2+\sin t}\,dt = \int_C\frac{2}{z^2+4iz-1}\,dz = 2\pi i\frac{1}{\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}.$$

Again the technique is fairly straightforward – even if it is quite length. And the important thing to note is that it allows us to find the integrals of many functions that we previously couldn't.

Things to know

- The definition of the Laurent series and how to find it for a standard example;
- The definition of an isolated singularity;
- The classification of isolated singularities as removable singularities, poles and essential singularities, and how to classify the singularities of a given function;
- The definition of a residue and how to find it for a given function, especially including applying l'Hopital's rule for simple poles;
- The statement and proof of Lemma 5.4;
- The statement of the Residue Theorem and how to use it to find the integral of a function;
- The application of the Residue Theorem to find integrals of real-valued integrals.

Problems

1. Find and classify the poles of the following functions and calculate the residue of each of them.

   (a) $\dfrac{z^2+2}{z-1}$

   (b) $\left(\dfrac{z}{2z+1}\right)^3$

   (c) $\dfrac{\exp z}{z^2+\pi^2}$

   (d) $\dfrac{a-z}{1-az}$ where $a \in \mathbb{R}$ and $a \neq 0$

2. Find and classify all the poles of $\dfrac{\exp z + 1}{\exp z - 1}$. For each of them find its residue.

3. Let C be the simple closed curve consisting of the line segment from -2 to 2 and then the semi-circle in the upper half-plane centred at 0 with radius 2. Use the Residue Theorem to find
$$\int_C \frac{1}{1+z^6}\,dz.$$

4. Suppose $f(z)$ and $g(z)$ both have a simple pole at a. Prove that $f(z)/g(z)$ has a removable singularity at a and hence can be extended to a holomorphic function in a neighbourhood of a.

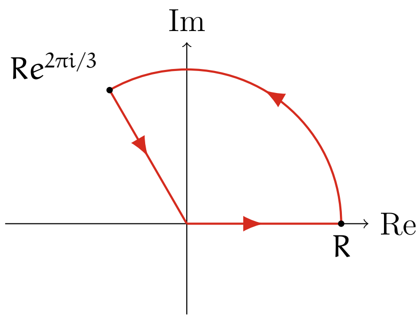
5. If $f(z)$ is holomorphic in Ω except at poles a_i, $i = 1, 2, \ldots, N$ of order m_i. Show that
$$f(z) = g(z)\prod_{i=1}^{N}\frac{1}{(z-a_i)^{m_i}}$$
where $g(z)$ can be extended to a holomorphic function in Ω.

6. Use residues to find

   (a) $\displaystyle\int_0^\infty \frac{1}{1+x^4}\,dx$. *Ans:* $\pi/(2\sqrt{2})$.

   (b) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$. *Ans:* $\pi/6$.

7. Use the curve shown below, where $R > 1$, to show that
$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}.$$


Im
$Re^{2\pi i/3}$
Re
R

Chapter 6 : Mapping Properties of Complex Functions

"The beauty of mathematics only shows itself to more patient followers."

Maryam Mirzakhani

6.1 Functions as mappings

Since we cannot visualise the graph of a complex-valued function, it is often easier to consider how a function maps shapes like rectangles or circles.

Example 30. Let $f(z) = z^2$.

If we consider the grid shown on the top-left of Figure 6.1. This is the square of side length $3/2$. In order to plot the mapping of this function we treat it as a function from $\mathbb{R}^2$ to itself:

$$(x + iy)^2 \mapsto x^2 - y^2 + 2xyi.$$

We parameterise the line from $(0, 0)$ to $(3/2, 0)$ by

$$(x, y) = (3/2t, 0), \qquad 0 \le t \le 1$$

then the mapping z^2 maps this to

$$(9t^2/4, 0), \qquad 0 \le t \le 1.$$

This can be done for all grid lines.

Things to notice in this example include the fact that the mapping takes the grid in the first quadrant and spins it around into the first and second quadrant. This is because if you think of it in polar form:

$$re^{i\theta} \mapsto r^2 e^{2i\theta}$$

then you can see that the argument is being doubled under this mapping whilst the absolute value is being squared. A sector, for example, like the one at the bottom of Figure 6.1, is mapped onto a sector twice the angle.

In general we often visualise complex functions as mappings of $\mathbb{C}$ to $\mathbb{C}$ in this way.

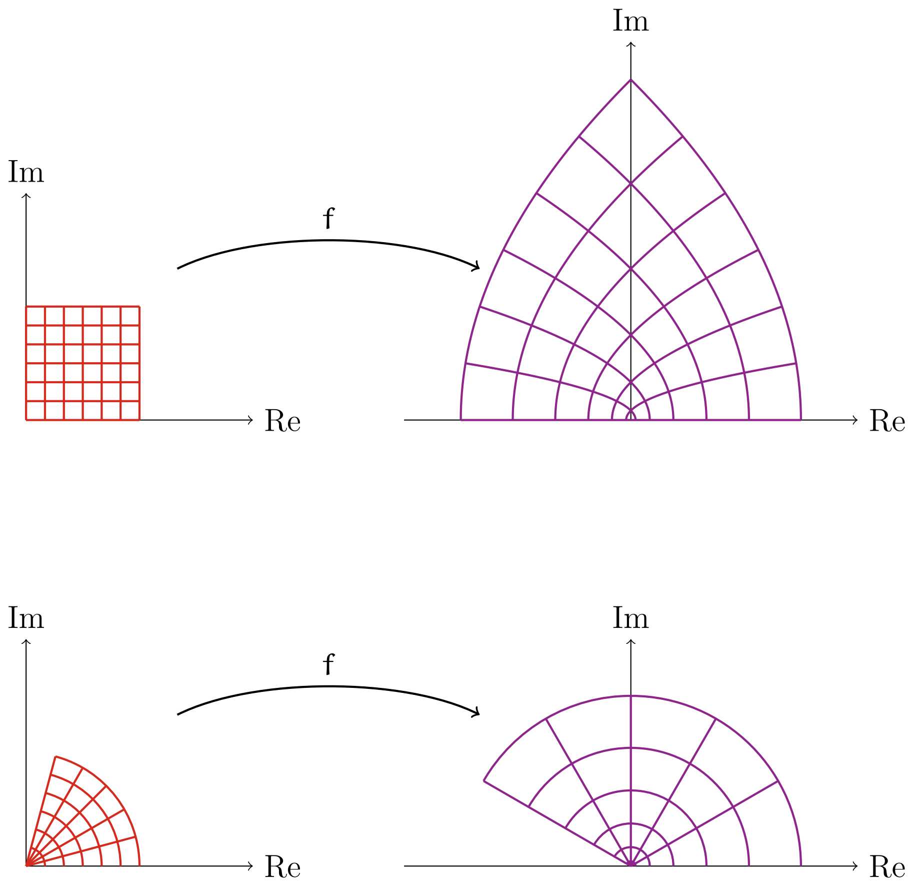


Figure 6.1: The mapping $f(z) = z^2$ on a grid and a sector.

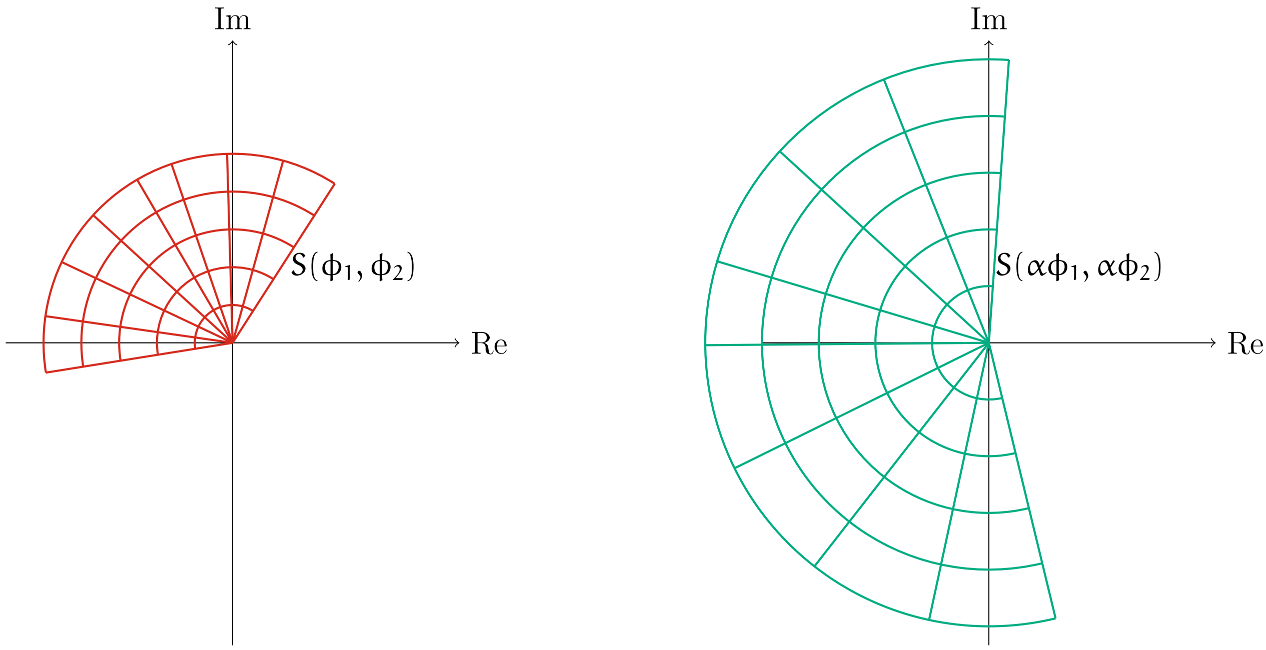


Figure 6.2: The mapping $f(z) = z^{\alpha}$ where $\alpha > 0$.

Suppose that $\alpha > 0$ is real. We saw in chapters 1 and 2 that the function $f(z) = z^{\alpha}$ is not in general single-valued in $\mathbb{C}$. A consequence of this is that we cannot define it easily. We can define it on a sector of opening less than $2\pi/\alpha$ at 0 though and in this domain it is holomorphic. In [1] Ahlfors describes this nicely on page 44 where he introduces the sectors $S(\phi_1, \phi_2) = \{z \in \mathbb{C} \colon \phi_1 < \arg z < \phi_2\}$. The mapping $f(z) = z^{\alpha}$ is injective as long as $0 \leq \phi_2 - \phi_2 \leq 2\pi/\alpha$. Then f maps this sector onto $S(\alpha\phi_1, \alpha\phi_2)$.

One thing to note here is that if $S(0, \phi)$ is a sector then the mapping

$$z \mapsto z^{\pi/\phi}$$

maps this sector onto $S(0, \pi)$ and this is the upper half-plane $\{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$. This can be useful for ironing out corners in domains.

The exponential function is relatively simple to understand. Recall that if $z = x + iy$ then

$$\exp z = e^{x} e^{iy}.$$

Thus if we consider a line with constant real part x_0, then this is mapped onto a circle of radius e^{x_0}. Similarly the line with constant imaginary part y_0 is mapped onto the ray with the argument, y_0. In Figure 6.3 the complex planes on the right show how the exponential function maps a rectangle

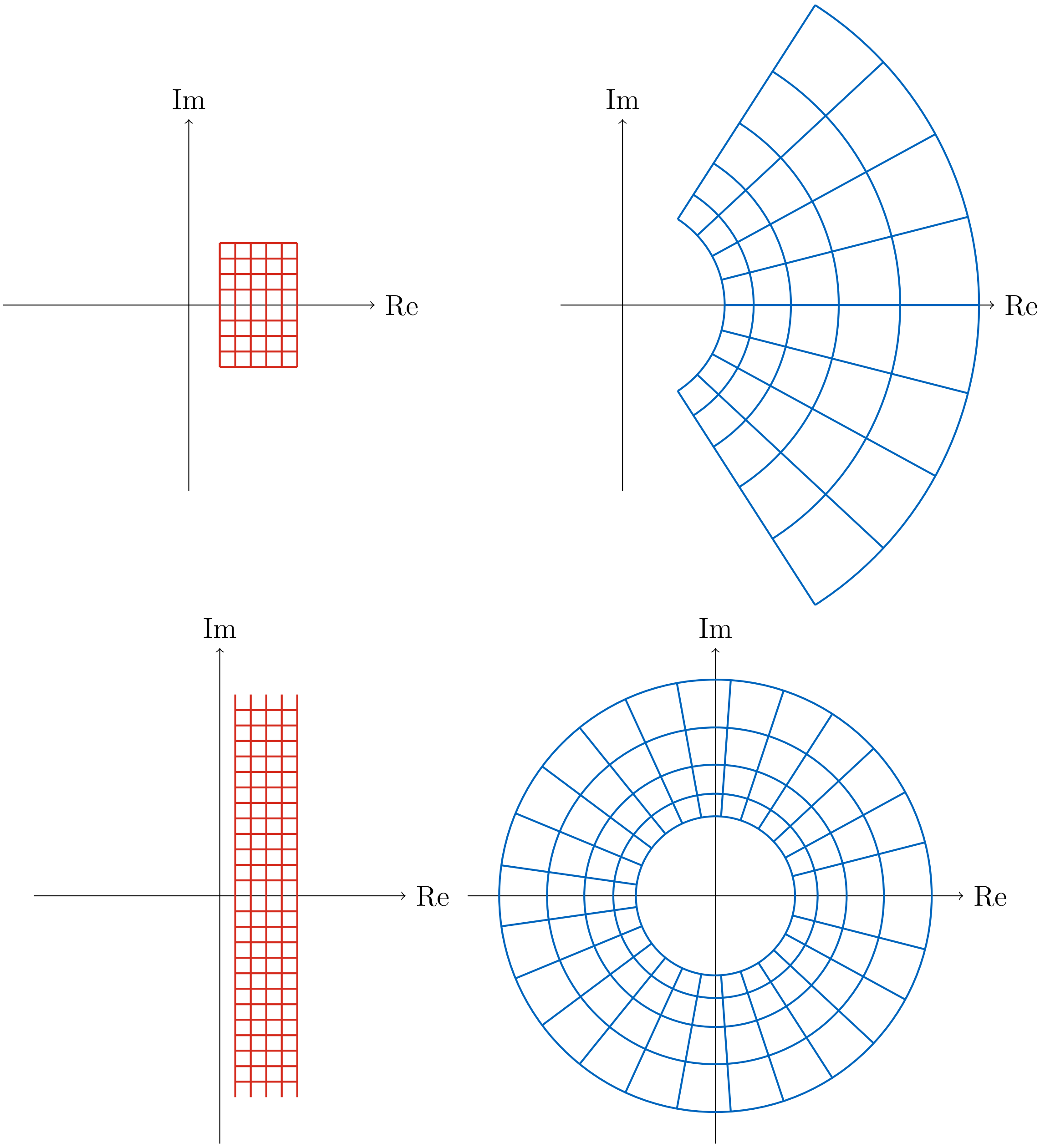


Figure 6.3: The mapping of the exponential function

and an infinite strip. Note that the top diagram in Figure 6.3 shows the rectangle being mapped *bijectively* onto the domain on the right. However in the bottom diagram the infinite strip is not bijectively mapped onto the annulus on the right.

Finally we will consider the mapping properties of the logarithm. Remember that the logarithm is not defined everywhere in a consistent manner. Instead we introduced tha α-logarithm:

$$\log_\alpha z = \log|z| + i\arg_\alpha z$$

where $\arg_\alpha z$ means the argument of z chosen in the range $[\alpha, \alpha + 2\pi)$.

Given a ϕ_1 and ϕ_2 with $0 \leq \phi_2 - \phi_1 \leq 2\pi$ we can find $\alpha \notin (\phi_1, \phi_2)$ such that $\log_\alpha$ maps the sector $S(\phi_1, \phi_2)$ bijectively onto the infinite strip

$$\{z \in \mathbb{C} \colon \phi_1 < \operatorname{Im} z < \phi_2\}.$$

As a particularly useful example $S(0, \pi)$ is the upper half-plane. This can be mapped by the principle logarithm (where $\alpha = 0$) onto the strip

$$\{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < \pi\}.$$

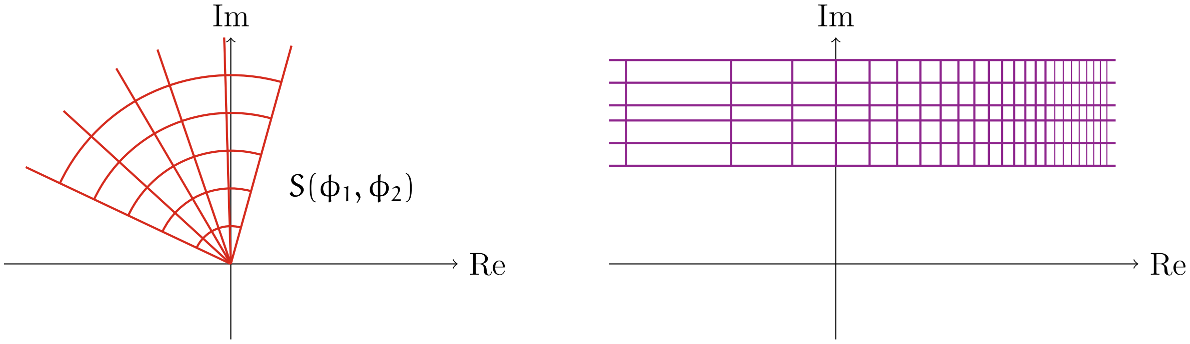


Figure 6.4: The mapping of the logarithm function

The mappings described above can be combined or modified to include mappings of many domains onto each other. For example the function

$$z \mapsto \operatorname{Log} z^2$$

maps the first quadrant in $\mathbb{C}$ onto the strip $\{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < \pi\}$ bijectively. The question of whether or not there is a bijective, holomorphic mapping from one domain onto another is a complicated one and the solution is remarkable and unique to complex analysis. It is the content of probably the most important result in complex analysis.

Theorem 6.1 – The Riemann mapping theorem

Suppose that Ω is a simply connected domain whose boundary consists of more than one point. Then there is a bijective, holomorphic mapping from Ω to the disk $B(0, 1)$.

Of course one can combine functions to find mappings from any such domain onto any other.

6.2 *The maximum modulus*

In calculus and real-analysis in general it is often useful to consider the problem of finding the maximum of a function in a domain. In complex analysis of course you can't maximise a function because $\mathbb{C}$ doesn't have an ordering, i.e. there is no notion of one complex number being larger than another. However we can maximise the real-valued function $|f(z)|$ in a domain and this is the problem we will look at in this section.

You've already seen one example of this when we looked at Liouville's theorem, Theorem 4.15. We can think of this theorem as saying that an entire function cannot map $\mathbb{C}$ into a bounded domain Ω.

We will begin with the following result that may take some time to digest.

Theorem 6.2 – Maximum modulus theorem

Suppose that $f(z)$ is a non-constant holomorphic function in a domain Ω. Then $|f(z)|$ does not attain its supremum in Ω.

If we let $M = \sup_{z\in\Omega} |f(z)|$ then this means that there is no point in Ω where $|f(z)| = M$. Remember that Ω is an open set so that this makes sense. To prove this important theorem we need the following.

Lemma 6.3 (Gauss' mean value theorem)**.** *Suppose* $f(z)$ *is holomorphic in a domain containing the closed disk* $\overline{B(a, r)} = \{z \in \mathbb{C} \colon |z - a| \leq r\}$. *Then*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt.$$

This looks suspiciously like Cauchy's integral formula. That is because it's simply a restatement. It should be noted then that this property holds for a much wider class of functions called *harmonic* functions and there the proof is more difficult.

Proof. We let $C = \{z \in \mathbb{C} \colon |z - a| = r\}$. We parameterise this by

$$p(t) = a + re^{it}, \qquad 0 \le t \le 2\pi.$$

Then by Cauchy's integral formula

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} f(p(t))p'(t)dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} ire^{it}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})dt \end{aligned}$$

□

We also need the following uniqueness result.

Lemma 6.4. *Suppose that* $f(z)$ *is holomorphic on a domain containing* $B(a,r)$ *and* $|f(z)| = m$ *for all* $z \in B(a,r)$*. Then* $f(z)$ *is constant throughout* Ω*.*

Proof. First if $m = 0$ then by Theorem 4.10 $f(z) = 0$ for all $z \in \Omega$. If $m > 0$ then since $f(z)$ is continuous there is a $\rho > 0$ such that $f(z) \neq 0$ for all $z \in B(a,\rho)$. So there is a branch of the logarithm for which $g(z) = \log_\alpha f(z)$ is holomorphic in $B(a,\rho)$. However

$$g(z) = \log|f(z)| + i\arg_\alpha z$$

and this function has constant real part. We have seen that such functions are constant throughout their domain of definition by the Cauchy-Riemann equations. So $f(z) = \exp g(z)$ is also constant. □

Proof of Theorem 6.2. We prove this by contradiction. Suppose that there is a point $a \in \Omega$ such that $|f(a)| \ge |f(z)|$ for all $z \in \Omega$. Let $R > 0$ be chosen so that $\overline{B(a,R)}$ is contained in Ω. For each $0 < r \le R$ we have, by Lemma 6.3,

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})dt.$$

Now $|f(z)|$ is not constant in $B(a,r)$ by Lemma 6.4 so we can find a $\rho < R$ and a $\phi \in [0,2\pi)$ such that $|f(a+\rho e^{i\phi})| < |f(a)|$. Since f is continuous, there is an interval I with $\phi \in I \subset [0,2\pi)$ such that

$$|f(a+\rho e^{it})| < |f(a)|, \qquad t \in I.$$

We therefore have

$$
\begin{aligned}
|f(a)| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(a+\rho e^{it})|dt = \frac{1}{2\pi}\left(\int_I |f(a+\rho e^{it})|dt + \int_{I^c} |f(a+\rho e^{it})|dt\right) \\
&< \frac{1}{2\pi}\int_I |f(a)|dt + \frac{1}{2\pi}\int_{I^c} |f(a)|dt, \qquad \text{where } I^c = [0,2\pi)\backslash I, \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(a)|dt = |f(a)|.
\end{aligned}
$$

This is of course a contradiction and so the theorem is proved. □

This is an important property of holomorphic functions. One consequence is that if $f(z)$ is holomorphic in a domain containing $\overline{B(0,r)}$ for all $r < R$ for some $R > 0$ then the maximum value of $|f(z)|$ in the disk $B(0,r)$ is taken at a point on the circle $\{z \in \mathbb{C} : |z| = r\}$. This justifies the following definition.

Definition 6.5 – The maximum modulus function

Suppose that $f(z)$ is a complex valued function. The *maximum modulus function* is

$$M(r;f) = \max_{\theta\in[0,2\pi)} |f(re^{i\theta})|.$$

By Theorem 6.2, if f is holomorphic in a domain containing $\overline{B(a,r)}$, the value of $M(r;f)$ is the maximum value taken by $|f(z)|$ in the whole of $B(0,r)$. Therefore we have that $M(r;f)$ is a strictly increasing function of r whenever f is a non-constant holomorphic function.

For non-constant entire functions we know by Liouville's theorem that $M(r;f) \to \infty$ as $r \to \infty$. For functions only holomorphic on a smaller domain we have the following.

Theorem 6.6 – Schwarz's lemma

Suppose that f is holomorphic on $B(0,1)$, $|f(z)| \le 1$ and $f(0) = 0$ then

$$M(r;f) \le r, \qquad 0 < r < 1.$$

If equality occurs at *any* value of r then $f(z) = \lambda z$ for some λ with $|\lambda| = 1$.

Proof. Since $f(0) = 0$ we can write $f(z) = \sum_{n=m}^{\infty} a_n z^n$ where $m > 0$ is the multiplicity of the zero at 0. In particular we can write

$$g(z) = \frac{f(z)}{z}$$

and $g(z)$ is holomorphic in $B(0, 1)$.

Now if $|z| \leq r < 1$ then by the maximum modulus theorem, Theorem 6.2,

$$|g(z)| \leq \max_{\theta \in [0,2\pi)} \frac{|f(re^{i\theta})|}{|re^{i\theta}|} \leq \frac{1}{r}.$$

This is true for all $r \in (0, 1)$ and so is also true if we let $r \to 1$. Therefore for all $z \in B(0, 1)$, $|g(z)| \leq 1$.

For $0 < r < 1$ it follows that

$$1 \geq \max_{\theta \in [0,2\pi)} |g(re^{i\theta})| = \max_{\theta \in [0,2\pi)} \frac{|f(re^{i\theta})|}{|re^{i\theta}|} = \frac{M(r; f)}{r}.$$

So $M(r; f) \leq r$.

To prove the second part suppose that $M(r; f) = r$ for some r, then $|g(z)| = 1$ for some $z \in B(0, 1)$. But this is the maximum value taken by $|g(z)|$ and so by Theorem 6.2 g must be constant. It follows that $f(z)/z = \lambda$ where $|\lambda| = 1$ as required. □

Thinking of f as a mapping, Schwarz's lemma says that if f maps $B(0, 1)$ into $B(0, 1)$ and $f(0) = 0$ then f maps $B(0, r)$ into $B(0, r)$ for any $0 < r < 1$. So f can be thought of as *contracting* the ball $B(0, r)$.

6.3 The argument principle

Suppose that C is a simple closed curve of finite length then a holomorphic function maps C onto another closed curve D of finite length. However it is no longer necessarily true that D is simple, it may wind its way around any number of times. For example the function $f(z) = z^2$ maps the circle $\{z \in \mathbb{C} : |z| = 1\}$ onto itself but the resulting curve is covered twice. In this section we prove the following result.

Theorem 6.7 – The argument principle

Let C be a simple closed curve of finite length and let $f(z)$ be a function holomorphic in a domain containing C except possibly at a set of isolated singularities. Then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$$

where N is the number of zeros and P the number of poles in the interior of C counted according to their multiplicity.

When we say that zeros or poles are *counted according to their multiplicity* in the statement above this means that if a zero, for example, has multiplicity 3 then it contributes 3 to the value of N.

Proof. Suppose a is a zero of order m. Then $f(z) = (z-a)^m g(z)$ where $g(z)$ has a removable singularity at a. Now

$$\frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1}g(z) + (z-a)^m g'(z)}{(z-a)^m g(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}.$$

Now if b is a pole of order n then $g(z) = (z-b)^{-n}h(z)$ and we may perform the same calculation to show that

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{-n}{z-b} + \frac{h'(z)}{h(z)}.$$

We can apply this calculation to all zeros and poles of $f(z)$. If $z_1, z_2, \ldots, z_k$ are the zeros of $f(z)$ interior to C and $a_1, a_2, \ldots, a_l$ are the poles then

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{k} \frac{m_i}{z - z_k} + \sum_{j=1}^{l} \frac{-p_j}{z - a_j} + \frac{G'(z)}{G(z)}.$$

Here $G(z)$ has no zeros or poles, m_i is the multiplicity of the zero z_i, and p_j the order of the pole at a_j.

Hence $f'(z)/f(z)$ has residues m_i at each z_i, and $-p_j$ at each a_j and by the Residue Theorem

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{k} m_i - \sum_{j=1}^{l} p_j = N - P.$$

□

The reason this is called the argument principle is the interpretation of the integral. The integrand $f'(z)/f(z)$ is the logarithmic derivative of $f(z)$, i.e. $\frac{d}{dz}\log f(z)$. Suppose that $f(z)$ is holomorphic and non-zero for all z in the interior of C. Then there is a branch of the logarithm for which $\log_\alpha f(z)$ is defined and is holomorphic. If C is parameterised by $p(t)$ for $a \le t \le b$ then

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)}dz &= \int_C \frac{d}{dz}\log_\alpha f(z)dz \\ &= \int_a^b p'(t)\frac{d}{dz}\log f(p(t))dt \\ &= \int_a^b \frac{d}{dt}\log f(p(t))dt \\ &= \left[\log|f(p(t))| + i\arg_\alpha f(p(t))\right]_a^b.\end{aligned}$$

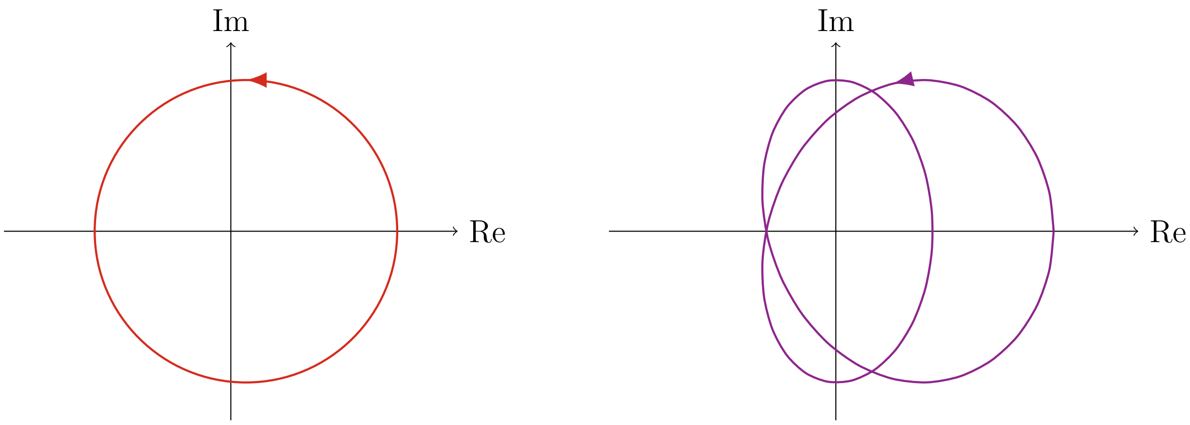


Figure 6.5: The mapping of the circle $\partial B(0.2, 2)$ under the mapping $z \mapsto z^2$

The curve C is closed so $|f(p(a))| = |f(p(b))|$. Thus the integral above is $i(\arg f(p(b)) - \arg f(p(a)))$. This is $2\pi i$ times the *winding number* of the curve $f(C)$. The winding number of $f(C)$ is the number of times a curve winds around the origin before closing. In Figure 6.5 the image of the circle on the left is the curve on the right which has winding number 2. This corresponds to the fact that the function $f(z) = z^2$ has two zeros. The winding number of a closed curve Γ is denoted by $n(\Gamma, 0)$ where we highlight 0 as the point around which Γ is winding. The argument principle is thus

$$n(f(C), 0) = N - P.$$

The classic application of the argument principle is the following important theorem.

Theorem 6.8 – Rouché's theorem

Let C be a simple closed curve in a domain Ω. Suppose that $f(z)$ and $g(z)$ are holomorphic in Ω and for all $z \in C$

$$|f(z) - g(z)| < |f(z)|.$$

Then $f(z)$ and $g(z)$ have the same number of zeros.

I will follow the proof given in [1, page 153] since it is an elegant, geometric proof. It is based on the observation that if Γ is contained in a disk that does not contain the origin then $n(\Gamma, 0) = 0$.

Proof. By assumption $f(z)$ and $g(z)$ are non-zero on C. Furthermore

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1, \qquad z \in C.$$

Therefore $F(z) = g(z)/f(z)$ maps C into the disk $B(1, 1)$. It follows that $n(F(C), 0) = 0$. Now

$$\frac{F'(z)}{F(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$$

so that

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_C \frac{F'(z)}{F(z)}dz = \frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}dz - \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz \\ &= N_g - N_f \end{aligned}$$

by the argument principle, where N_f and N_g are the number of zeros of $f(z)$ and $g(z)$. □

A continuous mapping does not necessarily map open sets to open sets. For example the real function $f(x) = x^2$ maps $(-1, 1)$ to $[0, 1)$. However for holomorphic functions this is an important and useful property. We can prove this as an application of Rouché's theorem.

Theorem 6.9 – The open mapping theorem

A non-constant holomorphic function maps open sets to open sets.

Such a function is called an *open mapping* – hence the name of the theorem. Recall that an open set is a set in which every point contains an open neighbourhood. So to prove that $f(A)$ is open when A is open we need to show that f maps open neighbourhoods of each point into another open neighbourhood.

Proof. Suppose $f \in \mathcal{H}(\Omega)$ for some domain Ω and let $z_0 \in \Omega$ be arbitrary. By Corollary 4.13 let $\delta > 0$ be chosen so that $B(z_0, \delta) \subset \Omega$ and there is only one zero of $f(z) - f(z_0)$ in $\overline{B(z_0, \delta)}$, namely z_0 itself.

Let ϵ be the minimum value of $|f(z) - f(z_0)|$ on the circle $\partial B(z_0, \delta) = \{z \in \mathbb{C} \colon |z - z_0| = \delta\}$. We will show that the image of $B(z_0, \delta)$ contains the disk $B(f(z_0), \epsilon)$.

Choose $w \in B(f(z_0), \epsilon)$. Then on $\partial B(z_0, \delta)$ we have

$$|f(z_0) - w| < \epsilon \leq |f(z) - f(z_0)|.$$

Therefore by Rouché's theorem

$$(f(z) - f(z_0)) + (f(z_0) - w) = f(z) - w$$

has the same number of zeros in $B(z_0, \delta)$ as $f(z) - f(z_0)$ does. Since $f(z) - f(z_0)$ has one zero (at $z = z_0$), the function $f(z) - w$ also has a zero. That is, $f(z) = w$ for some $z \in B(z_0, \delta)$. Since w was arbitrary the image of $B(z_0, \delta)$ must contain all the points of $B(f(z_0), \epsilon)$.

We have shown that every point in the image of Ω contains a neighbourhood, the image $f(\Omega)$ is also open. $\square$

Things to know

- The different behaviour of standard functions in terms of their mapping properties;
- The statement but not the proof of the Maximum Modulus Theorem and the definition of $M(r; f)$;
- The statement and proof of Gauss' mean value theorem, Lemma 6.3;
- The statement and proof of Schwarz's lemma;
- The statement of the argument principle in terms of winding numbers: $n(f(C), 0) = N - P$;
- The statement of Rouché's Theorem and the Open Mapping Theorem.

Problems

1. Let $f(z) = az + b$ be a linear function. Describe geometrically what this mapping does for different a and b (using terms like 'rotate' and 'shift' and 'scale').

2. In assignment 1 you showed that if $z \in \mathbb{C}$ satisfies $|z| < 1$ then
$$\operatorname{Re} \frac{1+z}{1-z} > 0.$$
What does this tell you about the mapping properties of $\frac{1+z}{1-z}$? What does this tell you about the mapping properties of $i\frac{1+z}{1-z}$? How about $\operatorname{Log} \frac{1+z}{1-z}$?

3. One application of Rouché's Theorem is to locate zeros of polynomials. Let $f(z) = z^7 - 4z^3 + z - 1$.

   (a) Define $g(z) = -4z^3$, so that $f(z) - g(z) = z^7 + z - 1$. Show that if $|z| = 1$ then
   $$|f(z) - g(z)| \leq 3;$$

   (b) deduce that for $|z| = 1$
   $$|f(z) - g(z)| < |g(z)|;$$

   (c) finally, apply Rouché's theorem to show that $p(z)$ has exactly three zeros in $B(0, 1)$, counting multiplicity.

4. Find the number of zeros, counting multiplicities, of $z^6 - 5z^4 + z^3 - 2z$ in $B(0, 1)$.

5. (Another proof of the fundamental theorem of algebra). Suppose $n \geq 1$ and
$$f(z) = a_0 + a_1 z + \cdots a_{n-1} z^{n-1} + a_n z^n.$$

   (a) Let $g(z) = a_n z^n$. Show that if $R = |z| > 1$ then
   $$|f(z) - g(z)| \leq M(n-1)R^{n-1}, \qquad M = \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\};$$

   (b) show that there is a $R > 0$ for which
   $$M(n-1)R^{n-1} < |a_n|R^n = |g(z)|;$$

   (c) deduce from Rounché's theorem that $f(z)$ has n roots.

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