MSO2120 Complete Notes

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So far in MSO2120 you have looked at analysis on the real line $\mathbb{R}$. We will now look at 'abstract' analysis: instead of the real numbers $\mathbb{R}$ we will look at an abstract set X.

Sets

Sets are regarded as the fundamental buildings blocks of mathematics. With a set X we can

- look at subsets,
- take unions, intersections, and complements,
- ask if an element is a member of the set.

Set Theory is the study of sets, and is particularly concerned with what sets are possible to construct, and the consequences of selecting different axioms.

Some examples of sets include

- three integers $\{4, 5, 7\}$,
- the real numbers $\mathbb{R}$, and
- the set of fruit $\{\text{bannana}, \text{cherry}, \text{apple}\}$

Sets with distance

In Mathematical Analysis we look at abstracts sets X with some additional structure: we are interested in sets with a notion of distance. Looking at the above examples

- three integers $\{4, 5, 7\}$: we might agree that the distance between 4 and 5 is 1 (these numbers are only 1 apart), and that the distance between 5 and 7 is 2.
- the real numbers $\mathbb{R}$: we might agree that the distance between two numbers x, y is $|x - y|$
- the set of fruit: what could we possibly mean by the distance between bannana and cherry?

We might describe a notion of distance for the fruit: for example, we might agree that the distance between bananna and cherry is 4, the distance between cherry and apple is 3 and the distance between bannana and apple is 6. The important thing is that we are **explicit** about what our notion of distance is.

The point about this course is that by studying these structures abstractly we can apply our findings to every set with a notion of distance, whether these sets are

- numbers,
- functions,
- matrices, or
- fruit

Structure we ignore

It often helps to have a concrete example in mind when looking at abstract objects. You've done a lot of work with the real numbers $\mathbb{R}$ and have used some of the other structures on this set.

- order: e.g. $3 < 4$
- vector space: e.g. $3 + 4 = 7$
- additive group: e.g. $3 - 3 = 0$
- multiplicative structure: e.g. $3 \times 4 = 12$

In Mathematical Analysis we are, at first, not interested in these other structures.

1 Metric Spaces

Definition 1.1: Metric

Let X be a set and $\mathrm{d}\colon X \times X \to \mathbb{R}$.
d is called a **metric** on X if

M1 $\mathrm{d}(x, y) \geq 0$ for all $x, y \in X$ (non-negativity)

M2 $\mathrm{d}(x, y) = 0$ if and only if $x = y$ (identity)

M3 $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ for all $x, y \in X$ (symmetry)

M4 $\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y)$ for all $x, y, z \in X$ (triangle inequality)

Definition 1.2: Metric spaces

Let X be a set and $\mathrm{d}\colon X \times X \to \mathbb{R}$ be a metric on X.
The pair (X, d) is called a **metric space**.

Example 1.3: The real line $\mathbb{R}$

Let $\mathrm{d}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\mathrm{d}(x, y) := |x - y| .$$

The pair $(\mathbb{R}, \mathrm{d})$ is a metric space.

Proof.

$$\mathrm{d}(x, y) = |x - y| \geq 0 \qquad \forall\, x, y, \in \mathbb{R}$$

so **M1** is satisfied.

Next, if $x = y$ then

$$\mathrm{d}(x, y) = \mathrm{d}(x, x) = |x - x| = |0| = 0 \qquad \forall\, x \in \mathbb{R}$$

so $x = y$ implies that $\mathrm{d}(x, y) = 0$.
Next, if $\mathrm{d}(x, y) = 0$ then

$$0 = \mathrm{d}(x, y) = |x - y|$$

which implies (Task 4) that $x - y = 0$ hence $x = y$. Consequently, $\mathrm{d}(x, y) = 0$ implies that $x = y$ so **M2** is satisfied.

Next,

$$\mathrm{d}(x, y) = |x - y| = |-(y - x)| = |-1|\,|y - x| = |y - x| = \mathrm{d}(y, x) \qquad \forall\, x, y, \in \mathbb{R}$$

so **M3** is satisfied.

Finally, (using Task 6)

$$\mathrm{d}(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = \mathrm{d}(x, z) + \mathrm{d}(z, y) \qquad \forall\, x, y, z \in \mathbb{R}$$

so **M4** is satisfied.

Consequently, d is a metric on $\mathbb{R}$, so (X, d) is a metric space. $\square$

Example 1.4: The set $\mathbb{R}^n$

Let $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\mathrm{d}_1(x,y) := \sum_{i=1}^{n} |x_i - y_i|,$$
$$\mathrm{d}_2(x,y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}},$$

and

$$\mathrm{d}_\infty(x,y) := \max_{i=1,\ldots,n} |x_i - y_i|$$

Each pair $(\mathbb{R}^n, \mathrm{d}_1)$, $(\mathbb{R}^n, \mathrm{d}_2)$ and $(\mathbb{R}^n, \mathrm{d}_\infty)$ is a metric space.

Proof. (Tasks 10-12) □

The metric d_2 is called the **Euclidean metric** on $\mathbb{R}^n$ and is our 'usual' notion of distance.

Example 1.5: The set $\mathbb{C}^n$

The set $\mathbb{C}^n$ consists of n-tuples of complex numbers
i.e. $z \in \mathbb{C}^n$ means that $z = (z_1, z_2, \ldots, z_n)$ with $z_i \in \mathbb{C}$ for $i = 1, \ldots, n$.
Let $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_\infty$ be the metrics from Example 1.4.
Each pair $(\mathbb{C}^n, \mathrm{d}_1)$, $(\mathbb{C}^n, \mathrm{d}_2)$ and $(\mathbb{C}^n, \mathrm{d}_\infty)$ is a metric space.

Proof. (Tasks 18-20) □

The metric d_2 is called the **Euclidean metric** on $\mathbb{C}^n$ and again is our 'usual' notion of distance.

Definition 1.6: Equivalent metrics

Let X be a set, and let d and ρ be metrics on X. The metrics d and ρ are **equivalent** if there exists a constant $C > 0$ such that

$$\frac{1}{C}\rho(x,y) \le \mathrm{d}(x,y) \le C\rho(x,y) \qquad \forall x, y \in X.$$

Lemma 1.7:

The metrics $\mathrm{d}_1, \mathrm{d}_2$ and d_∞ on $\mathbb{R}^n$ are equivalent.

Proof. We will show that $\mathrm{d}_\infty \le \mathrm{d}_1 \le \mathrm{d}_2 \le \sqrt{n}\mathrm{d}_\infty$

□

2 Open and closed sets

Let (X, d) be a metric space. Given a point $a \in X$ and a radius r we define two special subsets of X:

Definition 2.1: Balls

Let $a \in X$ and $r > 0$. We define

- the **open ball** with centre a and radius r is

$$B_r(a) := \{x \in X | \mathrm{d}(x, a) < r\} \subset X$$

- the **closed ball** with centre a and radius r is

$$B_r[a] := \{x \in X | \mathrm{d}(x, a) \leq r\} \subset X$$

Open sets

Definition 2.2: Open sets

A set $A \subset X$ is **open** if for each point $a \in A$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(a) \subset A$.

There is a potential confusion here: we've used the word **open** to define both the set $B_r(a)$ (Definition 2.1) and a property of sets (Definition 2.2). There's a good reason for this:

Lemma 2.3: Open balls are open

Let $a \in X$ and $r > 0$. The set $B_r(a)$ is open.

Proof.
Choose a point $x \in B_r(a)$.

We want to show that there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset B_r(a)$.

Now, as $x \in B_r(a)$ it follows from Definition 2.1 that $\mathrm{d}(x, a) < r$.

Let $\varepsilon = r - \mathrm{d}(x, a) > 0$ and observe that $\mathrm{d}(x, a) = r - \varepsilon$.

Now, if $z \in B_\varepsilon(x)$ then $\mathrm{d}(z, x) < \varepsilon$ so

$$\begin{aligned} \mathrm{d}(z, a) &\leq \mathrm{d}(z, x) + \mathrm{d}(x, a) && \text{(by **M4**)} \\ &< \varepsilon + \mathrm{d}(x, a) && (\text{as } \mathrm{d}(z, x) < \varepsilon) \\ &= \varepsilon + r - \varepsilon \\ &= r \end{aligned}$$

Consequently, $\mathrm{d}(z, a) < r$ which means that $z \in B_r(a)$.

As $z \in B_\varepsilon(x)$ was arbitrary it follows that

$$B_\varepsilon(x) \subset B_r(a).$$

As $x \in B_r(a)$ was arbitrary it follows that for all $x \in B_r(a)$ there exists an $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset B_r(a)$$

□

Open sets behave nicely under certain set operations:

Theorem 2.4: Properties of open sets

Let (X, d) be a metric space. Then

1. $\emptyset$ and X are open sets.
2. If $\{A_i\}_{i \in \mathcal{I}}$ is a (finite/countable/uncountable) family of open sets then
$$\bigcup_{i \in \mathcal{I}} A_i \qquad \text{is an open set}$$
3. If $A_1, \dots, A_k$ is a **finite** family of open sets then
$$\bigcap_{i=1}^{k} A_k \qquad \text{is an open set}$$

Proof. See Tasks 27-28. We need to adapt these arguments to infinite unions and to finite intersections:

□

It turns out that any open set can be thought of as a union of open balls:

Theorem 2.5:

A set is open if and only if it is the union of a collection of open balls.

Proof. See Tasks 29-30. □

Closed sets

Definition 2.6: Closed sets

A set $A \subset X$ is **closed** if its complement $A^c = \{x \in X \mid x \notin A\}$ is open.

Again, the potential inconsistency between closed balls (Definition 2.1) and closed sets (Definition 2.6) doesn't arise:

Lemma 2.7: Closed balls are closed

Let $a \in X$ and $r > 0$. The set $B_r[a]$ is closed.

Proof. See Task 33. □

Closed balls also behave nicely under certain set operations.

Theorem 2.8: Properties of closed sets

Let (X, d) be a metric space. Then

1. $\emptyset$ and X are closed sets.
2. If $\{A_i\}_{i\in\mathcal{I}}$ is a (finite/countable/uncountable) family of closed sets then
$$\bigcap_{i\in\mathcal{I}} A_i \qquad \text{is a closed set}$$
3. If $A_1, \ldots, A_k$ is a **finite** family of closed sets then
$$\bigcup_{i=1}^{k} A_k \qquad \text{is a closed set}$$

Proof. See Task 34. □

Important note:

The set operations in Theorem 2.8 are **different** to those of Theorem 2.4.
For open sets we had **infinite unions** and **finite intersections**.
For closed sets we had **finite unions** and **infinite intersections**.

Open sets under equivalent metrics

Lemma 2.9:

Let X be a set and let d and ρ be metrics on the set X. If d and ρ are equivalent metrics then a set $A \subset X$ is open with respect to d if and only if A is open with respect to ρ.

Proof. □

Corollary 2.10:

Let $A \subset \mathbb{R}^n$. The following statements are equivalent:

- A is open in $(\mathbb{R}^n, \mathrm{d}_2)$
- A is open in $(\mathbb{R}^n, \mathrm{d}_1)$
- A is open in $(\mathbb{R}^n, \mathrm{d}_\infty)$

Limit points

Definition 2.11: Limit points

Let (X, d) be a metric space, and let $A \subset X$.
A point $x \in X$ is a **limit point** of A iff every ball with centre x contains a point of A distinct from x.
The set of limit points of A is denoted A'.

Theorem 2.12: An equivalent definition of closed sets

A set A is closed if and only if A contains all of its limit points.

Closure

Definition 2.13: Closure

Let (X, d) be a metric space, and let $A \subset X$.
The **closure** of A, denoted $\bar{A}$, is the intersection of all closed sets containing A, i.e.,

$$\bar{A} = \bigcap_{\substack{A \subset K \\ K \text{closed}}} K$$

Theorem 2.14:

Let (X, d) be a metric space, and let $A \subset X$.
The closure of A is equal to the union of A and its limit points, i.e.

$$\bar{A} = A \cup A'$$

3 Convergence and Continuity

Definition 3.1: Convergence

Let (X, d) be a metric space. A sequence of points $\{x_n\}_{n\in\mathbb{N}} \subset X$ **converges** to a point $x \in X$ if for all $\varepsilon > 0$ there exists an $N > 0$ such that

$$n > N \quad \Rightarrow \quad \mathrm{d}(x_n, x) < \varepsilon.$$

If $\{x_n\}_{n\in\mathbb{N}}$ converges to x we may write

- $x_n \to x$, or
- $\lim_{n\to\infty} x_n = x$.

Convergence gives us a useful way of looking at limit points.

Theorem 3.2:

Let (X, d) be a metric space and $A \subset X$ be a set.

A point $x \in X$ is a limit point of A $\Longleftrightarrow$ there exist a sequence $\{x_n\}$ such that

- $x_n \in A$
- $x_n \neq x$
- $x_n \to x$.

Proof.

$\Leftarrow$
If $x_n \to x$ then for every $\varepsilon > 0$ there exists an $x_n \in B_\varepsilon(x)$. Consequently, every ball around x contains a point $x_n \in A$ that is distinct from x. Hence x is a limit point of A.

$\Rightarrow$
If x is a limit point of A then every ball centred on x contains a point of A distinct from x. Consequently, for each $n \in \mathbb{N}$ the ball $B_{1/n}(x)$ contains a point $x_n \in A$ with $x_n \neq x$.
Finally, as $\mathrm{d}(x_n, x) < \frac{1}{n} \to 0$ as $n \to \infty$, it follows that $x_n \to x$. $\square$

Which gives us a powerful way of understanding closed sets:

Theorem 3.3:

Let (X, d) be a metric spaces and let $A \subset X$ be a set.
A is closed $\iff$ if $\{x_n\} \subset A$ is a convergent sequence, then $\lim_{n\to\infty} x_n \in A$.

Proof.

$\Leftarrow$
We will show that A is closed by proving that A contains all of its limit points and applying Theorem 2.12.

Let $x \in X$ be a limit point of A. From Theorem 3.2 there exists a sequence $\{x_n\}$ such that $x_n \in A$ and $x_n \to x$.
As $\{x_n\}$ is a convergent sequence it follows that $\lim_{n\to\infty} x_n = x \in A$.
Hence A contains all of its limit points, hence A is closed.

$\Rightarrow$
Let $\{x_n\} \subset A$ be a convergent sequence, and let $\lim_{n\to\infty} x_n = x$.
If $x_n = x$ for some n, then $x \in A$, hence $\lim_{n\to\infty} x_n \in A$ as required.
Suppose that $x_n \neq x$ for all $n \in \mathbb{N}$. Then $x_n \in A$, $x_n \neq x$ and $x_n \to x$ so from Theorem 3.2 x is a limit point of A.
But as A is closed it follows from Theorem 2.12 that A contains all of its limit points, hence $x \in A$, as required.

□

Corollary 3.4:

A closed ball is closed.

We have already proved this in Lemma 2.7 directly from the definition of a closed set. This is an alternative proof that uses our more refined understanding of closed sets.

Proof. Let $B_r[a]$ be a closed ball.

Let $\{x_n\} \subset B_r[a]$ be a convergent sequence and let $\lim_{n\to\infty} x_n = x$. We want to show that $x \in B_r[a]$.

Suppose for a contradiction that $x \notin B_r[a]$, so $\mathrm{d}(x, a) > r$.

See Task 44. □

Definition 3.5: Continuity

Let (X, d) and (Y, ρ) be metric spaces.
The map $T\colon (X, \mathrm{d}) \to (Y, \rho)$ is **continuous at** $x \in X$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mathrm{d}(x, y) < \delta \quad \Rightarrow \quad \rho(T(x), T(y)) < \varepsilon.$$

We say that T is continuous if T is continuous at each point $x \in X$.

Theorem 3.6: Sequential characterisation of continuity

Let $T\colon (X, \mathrm{d}) \to (Y, \rho)$.

$$\text{The map } T\colon (X,\mathrm{d}) \to (Y,\rho) \text{ is continuous at } x \in X$$

$$\Updownarrow$$

$$\text{If } x_n \text{ is a sequence converging to } x \text{ in } (X,\mathrm{d}) \text{ then } T(x_n) \text{ is a sequence converges to } T(x) \text{ in } (Y,\rho).$$

Proof.

$\Leftarrow$

Assume T is continuous and that $x_n \to x$ in (X, d).

Fix $\varepsilon > 0$.

Since T is continuous we can find $\delta > 0$ such that $\mathrm{d}(x_n, x) < \delta$ implies that $\rho(T(x_n), T(x)) < \varepsilon$.

Further, since x_n converges to x in (X, d) there exists N_δ such that $\mathrm{d}(x_n, x) < \delta$ for all $n \geq N_\delta$.

Consequently, for $n \geq N_\delta$, we have

$$\begin{aligned} & \mathrm{d}(x_n, x) < \delta \\ \text{and} \quad & \mathrm{d}(x_n, x) < \delta \quad \Rightarrow \quad \rho(T(x_n), T(x)) < \varepsilon \\ \text{hence} \quad & \rho(T(x_n), T(x)) < \varepsilon. \end{aligned}$$

Hence for all $\varepsilon > 0$ there exists an N such that $n \geq N$ implies that $\rho(T(x_n), T(x)) < \varepsilon$, which is precisely that $T(x_n) \to T(x)$ in (Y, ρ).

$\Rightarrow$
Assume that for every $x_n \to x$ the sequence $T(x_n) \to T(x)$.

Suppose for a contradiction that T is not continuous, so there exists an $\varepsilon > 0$ such that for any $\delta > 0$ there exists an $x \in X$ with

$$\mathrm{d}(x_n, x) < \delta \qquad \text{and} \qquad \rho(T(x_n), T(x)) \geq \varepsilon$$

Let $x_n \in X$ be a sequence with $\mathrm{d}(x_n, x) < \frac{1}{n}$ and $\rho(T(x_n), T(x)) \geq \varepsilon$.

Then $x_n \to x$ in (X, d) but $T(x_n)$ does not converges to $T(x)$ in (Y, ρ), which is a contradiction.

Hence T is continuous.

□

Definition 3.7: The inverse image

Let $f: X \to Y$ be a map, and let $A \subset Y$ be a set. The inverse image of A under f, denoted $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}$$

Note that the inverse image of a set is **always** defined. We do **not** require the map f to be invertible.

Inverse images give us yet another characterisation of continuity:

Theorem 3.8: Inverse image characterisation of continuity

Let $T\colon (X, \mathrm{d}) \to (Y, \rho)$. The following are equivalent

1) The map $T\colon (X, \mathrm{d}) \to (Y, \rho)$ is continuous
2) for all open $O \subset Y$ the inverse image $T^{-1}(O) \subset X$ is open
3) for all closed $C \subset Y$ the inverse image $T^{-1}(C) \subset X$ is closed

Proof.

2) $\Rightarrow$ 1)
Assume that the inverse image of any open set is open.

Let $x \in X$ and consider the open set $B_\varepsilon(T(x)) \subset Y$. Then

$$T^{-1}(B_\varepsilon(T(x))) \subset X$$

is an open set containing x.

Consequently, there exists a $\delta > 0$ such that $B_\delta(x) \subset T^{-1}(B_\varepsilon(T(x)))$.

Now

$$\begin{aligned} & \mathrm{d}(x, y) < \delta \\ \Rightarrow\quad & y \in B_\delta(x) \\ \Rightarrow\quad & y \in T^{-1}(B_\varepsilon(T(x))) \\ \Rightarrow\quad & T(y) \in B_\varepsilon(T(x)) \\ \Rightarrow\quad & \rho(T(x), T(y)) < \varepsilon \end{aligned}$$

consequently T is continuous at x. As x was arbitrary it follows that T is continuous.

1) ⇒ 2)
Assume that T is continuous.

Let A be an open subset of Y and let $x \in T^{-1}(A) \subset X$.

Since A is open there exists a $\varepsilon > 0$ with $B_\varepsilon(T(x)) \subset A$.

Now, as T is continuous there exists a $\delta > 0$ such that

$$\mathrm{d}(x, y) < \delta \quad \Rightarrow \quad \rho(T(x), T(y)) < \varepsilon$$

which is that

$$y \in B_\delta(x) \quad \Rightarrow \quad T(y) \in B_\varepsilon(T(x))$$

which is that

$$y \in B_\delta(x) \quad \Rightarrow \quad y \in T^{-1}(B_\varepsilon(T(x)))$$

hence

$$B_\delta(x) \subset T^{-1}(B_\varepsilon(T(x)))$$

hence

$$B_\delta(x) \subset T^{-1}(A).$$

So we have shown that for all $x \in T^{-1}(A) \subset X$ there exists a $\delta > 0$ such that $B_\delta(x) \subset T^{-1}(A)$.

Consequently $T^{-1}(A)$ is open.

2) ⇔ 3)
See Task 53. □

Special continuous maps

Definition 3.9: Isometry

Let (X, d) and (Y, ρ) be metric spaces.
The map $T\colon (X, \mathrm{d}) \to (Y, \rho)$ is an **isometry** if

$$\mathrm{d}(x, y) = \rho(T(x), T(y)) \qquad \forall x, y \in X$$

Definition 3.10: Homeomorphism

Let (X, d) and (Y, ρ) be metric spaces.
The map $T\colon (X, \mathrm{d}) \to (Y, \rho)$ is a **homeomorphism** if T is bijective and both T and T^{-1} are continuous.

Metric subspaces

Lemma 3.11:

Let (X, d) be a metric space. If $A \subset X$ then (A, d) is a metric space.

Proof. See Task 58. □

Definition 3.12: Metric subspace

If (X, d) is a metric spaces and $A \subset X$ then the metric space (A, d) is called a **metric subspace** of (X, d).

4 Compactness

Recall Definition 2.9:[1]

Definition 4.1: Subsequences

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. The sequence $\{y_k\}_{k=1}^{\infty}$ is a **subsequence** of $\{x_n\}$ if

- $y_k = x_{n_k}$ for all $k \in \mathbb{N}$, and
- $n_k < n_{k+1}$ for all $k \in \mathbb{N}$

For notational convenience we will often simply write $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$.

Definition 4.2: Compact set

Let (X, d) be a metric space.
A set $A \subset X$ is **compact** if for every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in A$ there is a subsequence $\{x_{n_k}\}$ that converges to some element of A.

So there are two ways that a set A can fail to be compact:

1. There is a sequence in A with **no** convergent subsequences.
2. There is a sequence in A with a subsequence that converges to a point $x \notin A$.

A compact set can be thought of as 'small' in the sense that there isn't enough space to fit a sequence without it 'bunching up'.

[1] we repeat this definition here so that the abstract portion of the module is self-contained

Examples of compact sets

Example 4.3:

Let $(\mathbb{R}, \mathrm{d})$ be the real line with the usual Euclidean metric.
The unit interval $[0, 1]$ is a compact set.

Proof. See Coursework 3. □

Example 4.4:

Let $\left(\mathbb{R}^2, \mathrm{d}\right)$ be the real line with the usual Euclidean metric.
The unit circle $\left\{(x, y)\ |x^2 + y^2 = 1\right\}$ is compact.

Alternative characterisation of compact sets

Definition 4.5: Open covers

Let (X, d) be a metric space and let $A \subset X$. A (finite/countable/uncountable) collection of open sets $\{U_i\}_{i\in\mathcal{I}}$ is an **open cover** of A if

$$A \subset \bigcup_{i\in\mathcal{I}} U_i$$

Theorem 4.6: An equivalent definition of compactness

Let (X, d) be a metric space.
A set $A \subset X$ is **compact** if and only if for **every** open cover $\{U_i\}_{i\in\mathcal{I}}$ of A there are finitely many $U_{i_1}, U_{i_2}, \ldots, U_{i_N}$ that form a cover of A.

Proof. See appendix. □

So if A is compact then no matter what open cover we are given

$$A \subset \bigcup_{i\in\mathcal{I}} U_i$$

we can make a particular choice of finitely many of these U_i and still cover A:

$$A \subset \bigcup_{n=1}^{N} U_{i_n}$$

Properties of compact sets

Definition 4.7:

A set A is bounded if there exists a point $x \in X$ and a number $r > 0$ such that $A \subset B_r(x)$

Theorem 4.8:

If A is compact then A is bounded and closed.

Proof. See Coursework 3. □

Theorem 4.9:

Suppose $K \subset X$ is compact. If $A \subset K$ and A is closed then A is compact.

Proof. See Coursework 3. □

Lemma 4.10: Convergent sequences have convergent subsequences

Let (X, d) be a metric space. If $\{x_n\}_{n=1}^{\infty} \subset X$ is a sequence converging to $x \in X$ then every subsequence of $\{x_n\}_{n=1}^{\infty}$ also converges to x.

Proof. Let $\{x_n\}_{n=1}^{\infty} \subset X$ be a sequence converging to $x \in X$, and let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence.

We want to show that x_{n_k} converges to x.

Fix $\varepsilon > 0$. As $\{x_n\}$ converges to x there exists an $N > 0$ such that

$$n > N \quad \Rightarrow \quad \mathrm{d}(x_n, x) < \varepsilon. \tag{1}$$

Now, let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$. As n_k is increasing then for some $K > 0$

$$k > K \quad \Rightarrow n_k > N$$

consequently, from (1), it follows that

$$k > K \quad \Rightarrow \mathrm{d}(x_{n_k}, x) < \varepsilon.$$

So, for all $\varepsilon > 0$ we have found a $K > 0$ such that

$$k > K \quad \Rightarrow \mathrm{d}(x_{n_k}, x) < \varepsilon$$

which is precisely that the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x. □

4.1 Mapping properties of compactness

Theorem 4.11:

Let (X, d) and (Y, ρ) be metric spaces and let $f\colon (X, \mathrm{d}) \to (Y, \rho)$ be a continuous map.
If $K \subset X$ is a compact set then its image

$$f(K) = \{f(x) \mid x \in K\} \subset Y$$

is a compact set.

Proof. Suppose that $f\colon (X, \mathrm{d}) \to (Y, \rho)$ is a continuous map, and $K \subset X$ is a compact set.

We want to show that $f(K)$ is a compact set. We will do this using the sequential characterisation of compactness.

Let $\{y_n\}_{n=1}^{\infty} \subset f(K)$ be a sequence.

From the definition of $f(K)$ for each y_n there exists an $x_n \in K$ such that $f(x_n) = y_n$.

Consequently, $\{x_n\}_{n=1}^{\infty}$ is a sequence in K. But K is compact, hence there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.
Let $\lim_{k\to\infty} x_{n_k} = x$.

Now, as f is continuous

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(x)$$

hence

$$\lim_{k\to\infty} y_{n_k} = f(x).$$

Consequently $\{y_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{y_n\}_{n=1}^{\infty}$. Hence $f(K)$ is compact.

□

We can summarise the mapping properties of continuous functions:

Important note:

If $f\colon (X, \mathrm{d}) \to (Y, \rho)$ is a continuous map then

- $f^{-1}(O) \subset X$ is open for all open $O \subset Y$.
- $f^{-1}(C) \subset X$ is closed for all closed $C \subset Y$.
- $f(K) \subset Y$ is compact for all compact $K \subset X$.

Note that openness and closedness is preserved by **inverse** images, while compactness is preserved by **forward** images.

4.2 Applications of Compactness

Definition 4.12: Uniform continuity

Let (X, d) and (Y, ρ) be metric spaces. A map $f\colon (X, \mathrm{d}) \to (Y, \rho)$ is **uniformly** continuous if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mathrm{d}(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon \qquad \forall x, y \in X.$$

Theorem 4.13: Compact domain implies uniform continuity

Let (X, d) and (Y, ρ) be metric spaces, and suppose that X is compact.
If $f\colon (X, \mathrm{d}) \to (Y, \rho)$ is continuous then it is **uniformly** continuous.

Proof. Let X be and let $f\colon (K, \mathrm{d}) \to (Y, \rho)$ be continuous.

Fix $\varepsilon > 0$. Since f is continuous, for every point $x \in X$ there exists a $\delta_x > 0$ such that

$$\mathrm{d}(x, y) < \delta_x \qquad \Rightarrow \qquad \rho(f(x), f(y)) < \frac{\varepsilon}{2} \tag{2}$$

(Note that these δ_x depend on the particular value of x).

For each $x \in X$ let $J_x = B_{\delta_x/2}(x)$.

Since $x \in J_x$ the collection of sets $\{J_x\}_{x\in X}$ is an open cover of X. But since X is compact it has a finite subcover:

$$X \subset \bigcup_{k=1}^{n} J_{x_k}$$

for some $x_1, x_2, \ldots, x_n \in X$.

Define

$$\delta = \frac{1}{2}\min\{\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_n}\}.$$

As we are taking the minimum of finitely many positive values it follows that $\delta > 0$.

Now, let $x, y \in X$ and $\mathrm{d}(x, y) < \delta$.

Since $X \subset \bigcup_{k=1}^{n} J_{x_k}$ there exists a $k = 1, \ldots, n$ such that $x \in J_{x_k}$, hence

$$\mathrm{d}(x, x_k) < \frac{\delta_{x_k}}{2}. \tag{3}$$

Further

$$\begin{aligned} \mathrm{d}(y, x_k) &\le \mathrm{d}(x, x_k) + \mathrm{d}(x, y) \\ &\le \frac{\delta_{x_k}}{2} + \delta \\ &\le \delta_{x_k} \end{aligned} \tag{4}$$

Now from (3) and (2) it follows that

$$\rho(f(x), f(x_k)) < \frac{\varepsilon}{2}$$

and from (4) and (2) it follows that

$$\rho(f(y), f(x_k)) < \frac{\varepsilon}{2}$$

Consequently,

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_k)) + \rho(f(y), f(x_k))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mathrm{d}(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon \qquad \forall x, y \in X,$$

which is precisely that f is uniformly continuous. □

4.3 Compact sets in $\mathbb{R}^n$

In Coursework 3 you have demonstrated that, in a general metric space, a compact set is necessarily closed and bounded. We now prove that the converse holds **in Euclidean space**, which gives a nice characterisation of the compact sets.

We first prove the following:

Lemma 4.14:

Let $(\mathbb{R}^m, \mathrm{d})$ be the m-dimensional Euclidean space with the usual Euclidean metric.
Let $a, b \in \mathbb{R}$ with $a < b$. The cube

$$[a, b]^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^m \mid a \leq x_i \leq b \quad \forall\, i = 1, \ldots, m\}$$

is compact.

Proof. Recall that $[a, b] \subset \mathbb{R}$ is compact (see Task 74).

We proceed via induction on m.

Suppose that $[a, b]^m$ is compact. We want to show that $[a, b]^{m+1}$ is compact.

A point $x \in [a, b]^{m+1}$ has the form

$$(x_1, x_2, \ldots, x_m, x_{m+1})$$

which, for convenience, we can write as

$$(\hat{x}, x')$$

where $\hat{x} \in [a, b]^m$ and $x' \in [a, b]$.

Note that for $x, y \in [a, b]^{m+1}$

$$\mathrm{d}(x, y) = \sqrt{\sum_{i=1}^{m+1} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2 + |x_{m+1} - y_{m+1}|^2}$$
$$= \sqrt{\mathrm{d}(\hat{x}, \hat{y})^2 + \mathrm{d}(x', y')^2}$$

Now, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $[a, b]^{m+1}$.

Each $x_n = (\widehat{x_n}, x_n')$. Consider the sequence $\{\widehat{x_n}\}_{n=1}^{\infty} \subset [a, b]^m$. As $[a, b]^m$ is compact there is a subsequence $\{\widehat{x_{n_k}}\}_{k=1}^{\infty}$ that converges to some $\hat{x} \in [a, b]^m$.

Now consider the sequence $\{x'_{n_k}\}_{k=1}^{\infty} \subset [a, b]$. As $[a, b]$ is compact there is a subsequence $\left\{x'_{n_{k_l}}\right\}$ that converges to some $x' \in [a, b]$.

Further, as $\left\{\widehat{x_{n_{k_l}}}\right\}_{l=1}^{\infty}$ is a subsequence of the convergent sequence $\{\widehat{x_{n_k}}\}_{k=1}^{\infty}$ by Lemma 4.10 it also converges to $\hat{x}$.

Consequently, $\widehat{x_{n_{k_l}}} \to \hat{x}$ and $x_{n_{k_l}} \to x'$ as $l \to \infty$.

Write $x = (\hat{x}, x') \in [a, b]^{m+1}$ and observe that

$$d(x_{n_{k_l}}, x) = \sqrt{d\left(\widehat{x_{n_{k_l}}}, \hat{x}\right)^2 + d\left(x'_{n_{k_l}}, x'\right)^2} \to 0$$

as $l \to \infty$.

Hence $x_{n_{k_l}}$ converges to x.

Consequently, given a sequence $\{x_n\}_{n=1}^{\infty} \subset [a, b]^{m+1}$ we found a subsequence converging to a point $x \in [a, b]^{m+1}$, hence $[a, b]^{m+1}$ is compact.

□

Theorem 4.15: Heine-Borel

Let $(\mathbb{R}^m, d)$ be the m-dimensional Euclidean space with the usual Euclidean metric.
A set $K \subset \mathbb{R}^m$ is compact if and only if it is closed and bounded.

Proof. The "only if" direction is proved in Coursework 3.

Suppose $K \subset \mathbb{R}^m$ is closed and bounded. As K is bounded it is contained inside a ball $K \subset B_r(x)$ for some $r > 0$ and $x \in \mathbb{R}^m$. This ball is itself contained in a closed cube Q of side length $2r$ (can you find it explicitly?).

By Lemma 4.14 the cube Q is compact. Hence K is a closed subset of a compact set so is itself compact by Theorem 4.9. □

4.4 Compact sets in $C[0,1]$

Definition 4.16: Equicontinuity

A collection of functions $\mathcal{F} \subset C[0,1]$ is **equicontinuous** if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - y| < \delta \qquad \Rightarrow \qquad |f(x) - f(y)| < \varepsilon$$

for all $x, y \in [0,1]$ and all $f \in \mathcal{F}$.

Theorem 4.17: Arzela-Ascoli

Let $(C[0,1], \mathrm{d})$ be the set of continuous functions on $[0,1]$ with the usual metric

$$\mathrm{d}(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

A subset $\mathcal{F} \subset C[0,1]$ (i.e. a collection of functions) is compact if and only if

- $\mathcal{F}$ is bounded,
- $\mathcal{F}$ is closed, and
- $\mathcal{F}$ is equicontinuous.