MSO2120 Complete Tasks

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Properties of absolute value

▷ Task 1. What is the definition of the absolute value $|\cdot|$ on $\mathbb{R}$?

▷ Task 2. Show that $|x| \geq 0$ for all $x \in \mathbb{R}$

▷ Task 3. Show that $-|x| \leq x \leq |x|$ for all $x \in \mathbb{R}$.

▷ Task 4. Show that $|x| = 0$ if and only if $x = 0$.

▷ Task 5. Show that $|cx| = |c|\,|x|$ for all $c, x \in \mathbb{R}$

▷ Task 6. Prove the triangle inequality on $\mathbb{R}$:

$$|a + b| \leq |a| + |b| \qquad \forall\ a, b, \in \mathbb{R}$$

(hint: consider the inequality in Task 3)

Metric spaces

▷ Task 7. Let $X = \mathbb{R}$ and define

$$\mathrm{d}(x, y) := \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Is (X, d) a metric space?

▷ Task 8. Let X be a set and define

$$\mathrm{d}(x, y) := \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Is (X, d) a metric space?
(hint: in the previous task, where did you use the fact that $X = \mathbb{R}$?)

Cauchy-Schwarz inequality

▷ Task 9. Prove the Cauchy-Schwarz inequality

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$$

(hint: consider the expression $\sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2$)

More metric spaces

Consider the set $\mathbb{R}^n$.

▷ Task 10. Define

$$\mathrm{d}_2(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}$$

Show that $(\mathbb{R}^n, \mathrm{d}_2)$ is a metric space
(hint: use the Cauchy-Schwarz inequality)

▷ Task 11. Define

$$\mathrm{d}_1(x, y) := \sum_{i=1}^{n} |x_i - y_i|$$

Show that $(\mathbb{R}^n, \mathrm{d}_1)$ is a metric space

▷ Task 12. Define

$$\mathrm{d}_\infty(x, y) := \max_{i=1,\ldots,n} |x_i - y_i|$$

Show that $(\mathbb{R}^n, \mathrm{d}_\infty)$ is a metric space.

Visualising metrics

Consider the the set $\mathbb{R}^2$.

▷ Task 13. Sketch a graph of the points x satisfying $\mathrm{d}_2((0, 0), x) = 3$.
These are the points that are distance 3 from the origin $(0, 0)$ according to the metric d_2.

▷ Task 14. Sketch a graph of the points x satisfying $\mathrm{d}_1((1, 0), x) = 2$.
These are the points that are distance 2 from the point $(1, 0)$ according to the metric d_1.

▷ Task 15. Sketch a graph of the points x satisfying $\mathrm{d}_\infty((1, 2), x) = 4$.
These are the points that are distance 4 from the point $(1, 2)$ according to the metric d_∞.

▷ Task 16. Calculate all the distance between the points $(3, 0)$, $(0, 4)$ and $(0, 0)$ for each of the metrics $\mathrm{d}_1, \mathrm{d}_2$ and d_∞.
For which of these metrics does Pythagoras's Theorem hold?

Even more metric spaces

▷ Task 17. What is the definition of the absolute value $|\cdot|$ on $\mathbb{C}$?

▷ Task 18. Show that $(\mathbb{C}^n, \mathrm{d}_1)$ is a metric space.

▷ Task 19. Show that $(\mathbb{C}^n, \mathrm{d}_2)$ is a metric space.

▷ Task 20. Show that $(\mathbb{C}^n, \mathrm{d}_\infty)$ is a metric space.

Balls

▷ Task 21. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the usual Euclidean metric. Sketch the following sets, and write them in more familiar 'interval' notation:

(a) $B_1(3)$

(b) $B_2(4)$

(c) $B_3[7]$

Answer the following:

(d) is $2 \in B_1(3) \cup B_2(4)$?

(e) is $4 \in B_2(4) \cup B_3[7]$?

▷ Task 22. Consider the metric space $(\mathbb{R}, \mathrm{d})$ where d is the discrete metric. Explicitly find, and sketch, the following sets:

(a) $B_{0.5}(3)$

(b) $B_1(7)$

(c) $B_1[7]$

(d) $B_4(3)$

Open sets

▷ Task 23. Let $(\mathbb{R}, \mathrm{d})$ be the be the real line with the usual Euclidean metric.
Prove that the open interval $(a, b) \subset \mathbb{R}$ is an open set.

▷ Task 24. Show that $\{0\}$ is not an open set.

▷ Task 25. Give an example of a closed ball that is not open.

▷ Task 26. Give an example of a closed ball that is open
(hint: consider your answer to Task 22. (c))

Properties of open sets

Let (X, d) be a metric space.

▷ Task 27. Prove that the empty set $\emptyset$ is open, and that the entire space X is open.

▷ Task 28. Let $A, B \subset X$ be open sets. Prove that $A \cup B$ is open and that $A \cap B$ is open.

▷ Task 29. Why is the union of a collection of open balls open?

▷ Task 30. Let A be an open set. From the definition, this means that for each $a \in A$ there exists an open ball $B_\varepsilon(a) \subset A$. Can you write A as a collection of open balls?

Closed sets

▷ Task 31. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the usual Euclidean metric. Show that the following sets are closed:

(a) $(-\infty, -4] \cup [4, \infty)$

(b) $[a, b]$

(c) $\{0\}$.

▷ Task 32. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the discrete metric. Show that the following sets are closed:

(a) $\mathbb{R} \setminus \{7\}$

(b) $\{7\}$

(c) (a, b).

▷ Task 33. Let (X, d) be a metric space. Show that the closed ball $B_r[a]$ is closed.
(Hint: we wish to show that $B_r[a]^{\mathrm{c}}$ is open. Lemma 2.3 provides a template).

Properties of closed sets

▷ Task 34. Prove the following:

(a) $\emptyset$ and X are closed sets.

(b) If $\{A_i\}_{i\in\mathcal{I}}$ is a (finite/countable/uncountable) family of closed sets then

$$\bigcap_{i\in\mathcal{I}} A_i \qquad \text{is a closed set}$$

(c) If $A_1, \ldots, A_k$ is a **finite** family of closed sets then

$$\bigcup_{i=1}^{k} A_k \qquad \text{is a closed set}$$

(Hint: consider Theorem 2.4)

Equivalent metrics

▷ Task 35. Let $X = \mathbb{R}$. Show that the metrics d and ρ defined by

$$\mathrm{d}(x, y) := |x - y|$$

and

$$\rho(x, y) := \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

are **not** equivalent.

Limit points

▷ Task 36. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the usual Euclidean metric. Find the limit points of the following sets:

(a) $(1, 2)$

(b) $(1, 2] \cup \{0\}$

Write down a set $A \subset \mathbb{R}$ such that

(c) A contains all of its limit points.

(d) A doesn't contain all of its limit points.

▷ Task 37. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the discrete metric. Find the limit points of the following sets:

(a) $(1, 2)$

(b) $(1, 2] \cup \{0\}$

Does every set contain all of its limit points?

Closure

▷ Task 38. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the usual Euclidean metric. What is the closure of the set (a, b)?

▷ Task 39. Why is the closure of A closed?

▷ Task 40. Prove that A is closed if and only if $\bar{A} = A$.

Convergence

▷ Task 41. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the usual Euclidean metric. Show that the sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ converges to 0.

▷ Task 42. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the discrete metric. Show that the sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ doesn't converge to 0. Does this sequence converge?

▷ Task 43. Prove the following:

Let X be a set and δ, ρ be equivalent metrics on X. If a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x with respect to d, then the sequence converges to x with respect to ρ.

Convergence and Closedness

▷ Task 44. Let $\{x_n\} \subset B_r[a]$ be a sequence with limit $\lim_{n\to\infty} x_n = x$. Prove by contradiction that $\mathrm{d}(x, a) \leq r$

▷ Task 45. Let $(\mathbb{R}, \mathrm{d})$ be the real line with the usual Euclidean metric. Are the following sets closed?

(a) $\left\{x \in \mathbb{R} \,|x^2 + 2 \geq 0\right\}$

(b) $\left\{\frac{1}{n} \,|n \in \mathbb{N}\right\}$

(c) $\mathbb{N}$

(d) $\mathbb{Q}$

▷ Task 46. Consider the metric space $(\mathbb{R}, \mathrm{d})$ where d is the discrete metric.

(a) What interesting property do convergent sequences have in this space?

(b) Show that every subset of $(\mathbb{R}, \mathrm{d})$ is closed.

A metric space of functions

▷ Task 47. Let $C[0,1]$ be the set of continuous functions $f\colon [0,1] \to \mathbb{R}$. Define d by

$$\mathrm{d}(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

Show that $(C[0,1], \mathrm{d})$ is a metric space.

▷ Task 48. Let $f \in C[0,1]$ be defined by $f(x) = x^2$.

(a) Is $g(x) = x^3$ in the open ball $B_1(f)$?

(b) Find a function $h \in C[0,1]$ such that $h \notin B_1(f)$

▷ Task 49. Are the following sets closed?

(a) $\left\{f \in C[0,1] \mid f\left(\frac{1}{2}\right) = 3\right\}$

(b) $\{f \in C[0,1] \mid f(x) \geq \sin(x)\ \forall x \in [0,1]\}$

(c) $\{f \in C[0,1] \mid f(x) = qx \text{ where } q \in \mathbb{Q}\}$

(d) $\left\{f \in C[0,1] \mid f \text{ is differentiable at } x = \frac{1}{2}\right\}$
Hint: Consider the sequence f_n defined by

$$f_n(x) = \begin{cases} -x & x - \frac{1}{2} \leq -\frac{1}{n} \\ nx^2 & \frac{-1}{n} < x - \frac{1}{2} < \frac{1}{n} \\ x & x - \frac{1}{2} \geq \frac{1}{n} \end{cases}$$

Inverse images

▷ Task 50. Let $f\colon X \to Y$. Prove the following relationships about the inverse image of f:

(a) If $A, B \subset Y$ then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(b) If $A, B \subset Y$ then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(c) If $A, B \subset Y$ and $A \subset B$ then $f^{-1}(A) \subset f^{-1}(B)$

(d) If $A \subset Y$ then $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$.

Continuity

▷ Task 51. Let (X, d), (Y, ρ) and (Z, σ) be metric spaces. Suppose that the map $T\colon (X, \mathrm{d}) \to (Y, \rho)$ is continuous at $x \in X$ and the map $S\colon (Y, \rho) \to (Y, \sigma)$ is continuous at $T(x) \in Y$.
Show that the map

$$S \circ T\colon (X, \mathrm{d}) \to (Z, \sigma)$$

is continuous at $x \in X$.

▷ Task 52. Let (X, d) be a metric space and let $y \in X$ be a fixed element of X. Show that the map

$$T\colon (X, \mathrm{d}) \to (\mathbb{R}, \rho)$$

defined by

$$T(x) = \mathrm{d}(x, y)$$

is continuous, where ρ is the usual Euclidean metric.

▷ Task 53. Complete the proof of Theorem 3.8:
Let $T\colon (X, \mathrm{d}) \to (Y, \rho)$.
Show that the inverse image of open sets is open if and only if the inverse image of closed sets is closed.
Hint: consider the relation in **??** part (d). How does this help?

Continuous functions

▷ Task 54. Let $(\mathbb{R}, \mathrm{d})$ be the real line equipped with the discrete metric and let $(\mathbb{R}, \rho)$ be the real line equipped with the usual Euclidean metric.

(a) Prove that **every** map $f\colon (\mathbb{R}, \mathrm{d}) \to (\mathbb{R}, \rho)$ is continuous.

(b) Which maps $f\colon (\mathbb{R}, \rho) \to (\mathbb{R}, \mathrm{d})$ are continuous?

▷ Task 55. Give an explicit example of a continuous function $f\colon (X, \mathrm{d}) \to (Y, \rho)$ such that the image of an open set under f is **not** open.

▷ Task 56. Let $(C[0,1], \mathrm{d})$ be the set of continuous functions on $[0,1]$ equipped with the supremum metric

$$\mathrm{d}(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

and let $(\mathbb{R}, \rho)$ be the real line with the usual Euclidean metric.

Let $T\colon (C[0,1], \mathrm{d}) \to (\mathbb{R}, \rho)$ be defined by

$$T(f) = f\left(\tfrac{1}{2}\right).$$

(a) Show that T is continuous.

(b) Is T injective?

Isometries

▷ Task 57. Let $\left(\mathbb{R}^2, \mathrm{d}\right)$ be the real plane with the usual Euclidean metric.

(a) Fix $a \in \mathbb{R}^2$ and let $T\colon \left(\mathbb{R}^2, \mathrm{d}\right) \to \left(\mathbb{R}^2, \mathrm{d}\right)$ be defined by

$$T(x) = a + x$$

Show that T is an isometry.

(b) Fix $\theta \in \mathbb{R}$ and let $R\colon \left(\mathbb{R}^2, \mathrm{d}\right) \to \left(\mathbb{R}^2, \mathrm{d}\right)$ be defined by

$$R(x) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Show that R is an isometry.

▷ Task 58. Let (X, d) be a metric space and suppose that $A \subset X$. Show that (A, d) is a metric space.

▷ Task 59. Let (X, d) and (Y, ρ) be metric spaces. Let $T\colon (X, \mathrm{d}) \to (Y, \rho)$ be an isometry. Show that

(a) T is continuous,

(b) T is injective,

(c) The inverse $T^{-1}\colon (T(X), \rho) \to (X, \mathrm{d})$, is an isometry.

(d) Conclude that $T\colon (X, \mathrm{d}) \to (T(X), \rho)$ is a homeomorphism.

▷ Task 60. Write down an explicit example of a continuous bijective function $T\colon (X, \mathrm{d}) \to (Y, \rho)$ such that T^{-1} is **not** continuous.

(Hint: consider your answer to Task 54.)

Sequences

▷ Task 61. Let $(\mathbb{R}, \mathrm{d})$ be the real line equipped with the usual Euclidean metric. For each of the following write down a sequence $\{x_n\}$ that satisfies the given conditions:

(a) $x_n \to 1$

(b) x_n doesn't converge

(c) $x_n \in (0, 1)$ and $x_n \to 1$

(d) $|x_n - x_m| > \frac{1}{2}$ for all $n \neq m$.

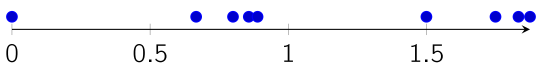
Subsequences

▷ Task 62. Let $(\mathbb{R}, \mathrm{d})$ be the real line equipped with the usual Euclidean metric. Consider the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} 1 - \frac{1}{n} & n \text{ odd} \\ 2 - \frac{1}{n} & n \text{ even} \end{cases}$$

(a) Sketch this sequence.

n	1	2	3	4	5	6	7	8	9
x_n	0	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{7}{4}$	$\frac{4}{5}$	$\frac{11}{6}$	$\frac{6}{7}$	$\frac{15}{8}$	$\frac{8}{9}$



(b) Write down a subsequence x_{n_k} such that $x_{n_k} \to 1$ as $k \to \infty$

(c) Write down a subsequence x_{n_k} such that $x_{n_k} \to 2$ as $k \to \infty$

▷ Task 63. Let $(\mathbb{R}, \mathrm{d})$ be the real line equipped with the usual Euclidean metric.
Write down a sequence that has **no** convergent subsequences.
hint: consider your answer to Task 61. d).

Compact sets

▷ Task 64. Let $(\mathbb{R}, \mathrm{d})$ be the real line equipped with the usual Euclidean metric.

Directly from the definition of compactness show that the following are **not** compact subsets of $(\mathbb{R}, \mathrm{d})$:

- the entire real line $\mathbb{R}$,
- the open interval $(0, 1)$,
- the rationals $\mathbb{Q}$

▷ Task 65. Show that the following collections of open sets form covers of the sets consider in **??**:

- $\{U_n\}_{n=1}^{\infty}$ where $U_n = (-n, n)$ covers the entire real line $\mathbb{R}$.
- $\{U_n\}_{n=1}^{\infty}$ where $U_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ covers the open interval $(0, 1)$

▷ Task 66. Hence show that $\mathbb{R}$ and $(0, 1)$ are **not** compact.

Properties of compactness

▷ Task 67. Using Theorem 4.8, why do we know that the sets from **??** are not compact?

▷ Task 68. Show that the closedness requirement of Theorem 4.9 is necessary (i.e. can you find a compact set K and a non-closed set $A \subset K$ such that A is **not** compact?).

Boundedness and Compactness: an illustrative example

Consider the set of infinite sequences in $\mathbb{R}$

$$X = \{(x_1, x_2, \ldots) \mid x_n \in \mathbb{R}\}$$

For example, the sequence $\mathbf{x} = (x_1, x_2, \ldots)$ defined by $x_n = \frac{1}{n}$ is an element of X.

The map $\mathrm{d}(\mathbf{x}, \mathbf{y}) = \max_{n \in \mathbb{N}} |x_n - y_n|$ is a metric on X (you can prove this if you like!) so (X, d) is a metric space.

Let $\mathbf{0} = (0, 0, 0, \ldots)$

▷ Task 69. Why is the set $B_1[\mathbf{0}]$ a closed, bounded set?

Consider the sequence $\{\mathbf{e}_j\}_{j=1}^{\infty} \subset X$ defined by

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, 0, 0, 0, \ldots) \\ \mathbf{e}_2 &= (0, 1, 0, 0, 0, \ldots) \\ \mathbf{e}_3 &= (0, 0, 1, 0, 0, \ldots) \\ \mathbf{e}_4 &= (0, 0, 0, 1, 0, \ldots) \\ &\vdots \end{aligned}$$

▷ Task 70. Show that $\mathbf{e}_j \in B_1[\mathbf{0}]$ for each $j = 1, \ldots, \infty$.

▷ Task 71. Show that $\mathrm{d}(\mathbf{e}_j, \mathbf{e}_k) = 1$ for $j \neq k$.

▷ Task 72. Why does the sequence $\{\mathbf{e}_j\}_{j=1}^{\infty}$ have no convergent subsequences?

▷ Task 73. Why is this example interesting in light of Theorem 4.8 ?

Mapping properties of continuous functions

▷ Task 74. Let $(\mathbb{R}, \mathrm{d})$ be the real line equipped with the usual Euclidean metric. Prove that each of the following sets is compact:

(a) the closed interval $[0, 4]$
(b) the closed interval $[3, 12]$
(c) an arbitrary closed interval $[a, b]$
(d) the circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subset of $(\mathbb{R}^2, \rho)$ where ρ is the usual Euclidean metric.

You may assume that the closed interval $[0, 1]$ is compact (as you've proved in the coursework).

▷ Task 75. Is there a continuous map $f\colon (\mathbb{R}, \mathrm{d}) \to (\mathbb{R}, \mathrm{d})$ such that $f([0, 1]) = \mathbb{R}$?

▷ Task 76. Provide counterexamples to the converses of the mapping properties of continuous functions (see the "Important note" on page 2 of Week 23 lecture notes).

Uniform convergence

▷ Task 77. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ (where the metrics are the usual Euclidean metrics).

Fix $x \in \mathbb{R}$. Show that f is continuous at x. Conclude that f is continuous.

Is f **uniformly** continuous?

▷ Task 78. Let $g\colon (0, 1) \to \mathbb{R}$ be defined by $g(x) = 1/x$ (where the metrics are the usual Euclidean metrics).

Fix $x \in \mathbb{R}$. Show that g is continuous at x. Conclude that g is continuous.

Is g **uniformly** continuous?

▷ Task 79. Show that the following maps are **uniformly** continuous. The metrics are the usual Euclidean metrics throughout:

- $f\colon [0, 1] \to \mathbb{R}$ defined by $f(x) = x^2$
- $g\colon [-13, 27] \to \mathbb{R}$ defined by $g(x) = x^{17} - x^4 + e^x - \sin(x)$
- $h\colon S^1 \to \mathbb{R}$ defined by $h(x, y) = x^2y^3$ where $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Heine-Borel

▷ Task 80. Explicitly write down the cube Q used in the proof of Theorem 4.15.

▷ Task 81. By following the proof of Lemma 4.14, show that the square $[a, b] \times [a, b] \subset \mathbb{R}^2$ is compact, simplifying the notation wherever you can.

▷ Task 82. Prove that the following sets are compact

- The line segment $\{(t, t) \in \mathbb{R}^2 \mid t \in [0, 1]\}$
- The union of discs
$$B_1[(2, 1)] \cup B_7[(4, 3)] \cup B_{\sqrt{2}}[(-2, 3)] \subset \mathbb{R}^2$$

Equicontinuity and Arzela-Ascoli

▷ Task 83. Let $f_1, f_2, \ldots, f_n \in C[0,1]$. Show that the **finite** family of functions

$$\mathcal{F} = \{f_j\}_{j=1}^{n}$$

is equicontinuous.

▷ Task 84. Let $(C[0,1], \mathrm{d})$ be the space of continuous functions on $[0,1]$ with the usual metric

$$\mathrm{d}(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

Show that $\mathcal{F} \subset C[0,1]$ is bounded if and only if there exists an $M > 0$ such that

$$|f(x)| \leq M$$

for all $x \in [0,1]$ and all $f \in \mathcal{F}$.

▷ Task 85. How does the Mean Value Theorem help us to find an appropriate δ in a proof of continuity? What condition on the derivative is sufficient to give uniform continuity? Using this fact, write down a set $\mathcal{F}$ of (infinitely many) functions such that $\mathcal{F}$ is equicontinuous.

▷ Task 86. Let $f_{a,b,c} \in C[0,1]$ be defined by

$$f_{a,b,c}(x) = ax^2 + bx + c.$$

Show that $\mathrm{d}(f_{a,b,c}, 0) \leq |a| + |b| + |c|$.

▷ Task 87. Let

$$\mathcal{F} = \{f_{a,b,c} \mid |a| + |b| + |c| \leq 17\}$$

Show that $\mathcal{F}$ is equicontinuous.

▷ Task 88. Hence show that there exists a convergent subsequence of $\mathcal{F}$.

