Chapter 1
On the Depth of Modular Invariant Rings for the Groups $C_p \times C_p$

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1.1 Introduction

Let G be a finite group acting linearly on a finite-dimensional vector space V over a field k. We assume that the characteristic of k is a prime number p which divides $|G|$, so that V is a *modular representation* of G. Define $R := S(V^*) := \mathrm{Sym}(V^*)$ to be the symmetric algebra over the dual space V^* with k - basis $x_1, \cdots, x_n$. Then R is isomorphic to the polynomial ring $k[x_1, \cdots, x_n]$, on which G acts naturally by graded algebra automorphisms, extending the linear action of G on V^*. We usually adopt the convention of writing V as a left kG - module and V^* as a right module with $\lambda g = \lambda \circ g$ for $\lambda \in V^*$ and $g \in G$. The *ring of invariants* or *invariant ring* R^G is defined as follows:

Definition 1.

$$R^G := \{r \in R : rg = r \ \forall g \in G\}$$

If $\mathrm{char}(k)$ divides $|G|$, we will call R^G a *modular* ring of invariants, if $\mathrm{char}(k)$ does not divide $|G|$, R^G will be called *non - modular*.

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For example, if $G = \Sigma_n$ and V is the standard permutation representation, then $R = k[e_1, \cdots, e_n]$ is a polynomial ring generated by the elementary symmetric polynomials $e_1, \cdots, e_n$. In the general case, R is always a finitely generated k - algebra, due to a classical result of Emmy Noether. However, it will not necessarily be a polynomial ring. It is well known that in the non - modular case, R^G is a polynomial ring if and only if the image of G in $\mathrm{GL}(V)$ is generated by reflections. In the modular case, a classification of representations with R^G being a polynomial ring is not known, except for p - groups and representations over the prime field $\mathbb{F}_p$. It is known however, that R^G being a polynomial ring is a rare condition. For these statements and more general background on invariant theory we refer the reader to the excellent monographs [2], [6] and [18].

From "Noether normalisation", another classical result of Emmy Noether, we know that there is a subset of homogenous elements $\{y_1, \cdots, y_n\} \subseteq R^G$, generating a polynomial subring $\mathscr{P} := k[y_1, \cdots, y_n] \leq R^G$, such that R^G is a finitely generated $\mathscr{P}$ - module. Such a subset $\{y_1, \cdots, y_n\}$ is called a **homogeneous system of parameters (hsop)**. If R^G is a free module over $\mathscr{P}$, then R^G is a **Cohen - Macaulay** ring. The technical definition of Cohen - Macaulay rings is in terms of *regular sequences* which we will now give for graded connected k - algebras, i.e. finitely generated $\mathbb{N}_0$ graded k algebras whose degree zero component is k. Let $A = \oplus_{i \in \mathbb{N}_0} A_i$ be such a graded connected k - algebra with $A^+ := \oplus_{i>0} A_i$; furthermore let $J \subseteq A^+$ be a homogeneous ideal and M be a graded A-module. A sequence of homogeneous elements $(a_1, \cdots, a_k)$ with $a_i \in J$ is called M-**regular**, if for every $i = 1, \cdots, k$ the multiplication by a_i induces an injective map on the quotient ring $M/(a_1, \cdots, a_{i-1})M$. It is known that all maximal M-regular sequences in J have the same length $\mathrm{grade}(J, M)$, called the grade of J on M and one defines

$$\mathrm{depth}(A) := \mathrm{grade}(A^+, A).$$

It is a theorem in commutative algebra that A is Cohen - Macaulay if and only if $\mathrm{depth}(A) = \mathrm{Dim}(A)$, where $\mathrm{Dim}(A)$ denotes the Krull dimension of the ring A.

Eagon and Hochster [7] showed that every non-modular invariant ring of a finite group is Cohen-Macaulay, but this does not remain true in the modular case. Determining the depth of modular rings of invariants remains an important problem but often is a very difficult challenge. The well known Auslander-Buchsbaum formula tells us that the difference between the depth and Krull dimension of R^G is equal to its projective dimension as a module over an hsop, so the depth can be viewed as a measure of structural complexity of R^G, telling us "how close" it is to being a free module over a polynomial ring.

In this paper we are interested in calculating the *depth* of certain types of rings of invariants. For more details and results regarding depth, the reader is referred to [5]. Thanks to Ellingsrud and Skjelbred [9], we do have a lower bound for the depth of an invariant ring. Their result was strengthened in [13] to the following:

Theorem 1. *Let G be a finite group and k a field of characteristic p. Let P be a Sylow-p-subgroup and let V be a left kG-module. Let R denote the symmetric algebra $S(V^*)$ which has a natural right-module structure. Then*

$$\operatorname{depth}(R^G) \geq \min\{\dim(V), \dim(V^P) + cc_G(R) + 1\}$$

where V^P denotes the fixed point space of P on V and $cc_G(R)$ is called the cohomological connectivity *and defined as* $\min\{i > 0 : H^i(G,R) \neq 0\}$.

A representation V for which this inequality is in fact an equality is called *flat*. Since for $R = S(V^*)$ the Krull - dimensions $\operatorname{Dim}(R^G)$ and $\operatorname{Dim}(R)$ coincide, both being equal to $\dim V$, it is clear that if $\dim(V^P) + cc_G(R) + 1 \geq \dim(V)$, then V is flat and R^G is Cohen-Macaulay. Such a representation will be called *trivially flat* in this article.

At this point we would like to thank an anonymous referee for helpful comments and suggestions.

1.2 Flatness and Strong Flatness

In [13], the authors not only proved the inequality in Theorem 1, but also obtained necessary and sufficient geometric conditions for it to actually be an equality. We briefly sketch the results there:

Theorem 2. *Let V be a kG-module, $R := S(V^*)$, and $m := cc_G(R)$ where we assume $m < \operatorname{codim}(V^P) - 1$. Then the following are equivalent:*

1. $\mathfrak{i}_G = \sqrt{\operatorname{Ann}_{R^G}(\alpha)}$ *for some* $0 \neq \alpha \in H^m(G,R)$*;*
2. $\operatorname{grade}(\mathfrak{i}_G, H^m(G,R)) = 0$*;*
3. $\operatorname{grade}(\mathfrak{i}_G, R^G)$ = *m+1;*
4. V is flat, i.e $\operatorname{depth}(R^G) := \operatorname{grade}(R^G_+, R^G) = \dim(V^P) + m + 1$.

In the above, $\mathfrak{i}_G$ is the prime ideal of R^G defined as $\mathfrak{i}_G := R^G \cap \mathfrak{i}_P$, where $\mathfrak{i}_P := \mathfrak{I}(V^P)$ is the prime ideal of R generated by all linear forms in V^* which vanish on the P - fixed point space V^P. The equivalence 3. $\Leftrightarrow$ 4. follows from the formula

$$\operatorname{depth}(R^G) = \dim(R/\mathfrak{i}) + \operatorname{grade}(\mathfrak{i}_G, R^G)$$

proved in [14]. The implication (2) $\Leftrightarrow$ (3) is more subtle, and follows from studying a certain spectral sequence. For details, see [13], chapter 7.

We want to use condition 1. to identify flat representations. In our attempt to do this, we make a further simplification. Consider the following ideal of R^G

$$I := \sum_{N<P} \operatorname{Tr}^G_N(R^N)$$

where the sum runs over all maximal subgroups of P. Here Tr^G_N denotes the *transfer homomorphism* $\operatorname{Tr}^G_N : R^N \to R^G$

$$\operatorname{Tr}^G_N r = \sum_{g \in S} rg$$

where S is a set of right coset representatives for N in G.

The ideal I above is called the *Relative Transfer Ideal*, and has been studied widely in connection with modular invariant theory. For example it is known that the quotient ring $R^G/\sqrt{I}$ is always Cohen-Macaulay ([12]). So there is a sense in which I contains the extra complexity in a modular ring of invariants. It turns out (see [13], chapter 3) that $\mathrm{i}_G = \sqrt{I}$. Since for any ideal J and ring A one has $\mathrm{grade}(J,A) = \mathrm{grade}(\sqrt{J},A)$ (see [5]), we now have necessary and sufficient conditions for flatness which we can hope to exploit.

Theorem 3. *Let V be a kG-module and $R := S(V^*)$. Let $m := cc_G(R)$. Then the following are equivalent:*

1. *V is flat*
2. *Either $m+1 \geq \mathrm{codim}(V^P)$ or there exists $0 \neq \alpha \in H^m(G,R)$ which is annihilated by every element of the relative transfer ideal.*

In practice it is difficult to find such an α, unless one knows explicitly the action of R^G on $H^m(G,R)$. Since finding the indecomposable summands of R as a kG - module is an unsolved problem, it is clear that this method is not really tractable. For this reason, we introduce a narrower class of representations:

Theorem 4. *Let V be a finite kG-module, and suppose that $0 \neq \tau \in H^m(G,R)$, where $m = cc_G(R)$ is a cohomology class such that*

$$\mathrm{res}^P_N(\tau) = 0$$

for each maximal subgroup $N < P$. Then V is flat.

Remark: This result is based on [13], Theorem 2.6.

Proof. Note that $H^m(G,R)$ is a direct summand of $H^m(P,R)$, with the restriction map res^G_P an injective map $H^m(G,R) \to H^m(P,R)$. So we may identify $H^m(G,R)$ with the image of res^G_P, and τ satisfying the conditions of the theorem satisfies $\mathrm{res}^G_N(\tau) = 0$ for all $N < P$. By [17], Lemma 1.3, elements in the image of the relative transfer Tr^G_N annihilate those in the kernel of the restriction $\ker \mathrm{res}^G_N$. Thus elements $\tau \in H^m(G,R)$ satisfying the conditions of the theorem are annihilated by every element of I. The result now follows from Theorem 3.

If G is a p - group with $p = \mathrm{char}(k)$, then $H^1(G,k) \neq 0$ and, since $k \cong R_0$, we get $H^1(G,R) \neq 0$, so $cc_G(R) = 1$. We shall call a representation V *strongly flat*, if $cc_G(R) = 1$ and if there exists a cohomology class $0 \neq \tau \in H^1(G,R)$ with $\mathrm{res}^P_N(\tau) = 0$ for every maximal subgroup $N < P$. Note that $H^1(G,R)$ has a direct summand isomorphic to $H^1(G,V^*)$; if we can find a suitable $0 \neq \tau$ satisfying the above condition in this direct summand, then the representation will be called *linearly flat*.

As the name suggests, not every representation which is flat is strongly so - we show this by example in section 5. However, strong flatness is a sufficiently general notion to give us plenty of new examples of flat representations. It also has the following two desirable properties not shared by mere flatness:

Lemma 1. *Strong flatness has the following properties:*

1. *A representation V is strongly flat if and only if there is a direct summand W^* (as a G-module) of $S(V^*)$ for which W is linearly flat.*
2. *If U, V are kG-modules and U is strongly flat, then so is $U \oplus V$.*

Proof. The first property is a consequence of the splitting

$$H^m(G,R) = \bigoplus_{i \geq 0} (H^m(G,S^i(V^*)))$$

To prove the second property, use the formula $S(U \oplus V) = S(U) \otimes S(V)$. Now $k = S^0(V)$ is a direct summand of $S(V)$, and so $S(U) \otimes k \cong S(U)$ is a direct summand of $S(U \oplus V)$. The result now follows from the first property.

If the trivial kG-module k is strongly flat, then every kG-module is strongly flat, because for each kG-module V we have that $R_0 = k$. From this we obtain immediately the result of Ellingsrud and Skjelbred that every modular representation of a cyclic group of prime order is flat - in this case there are no nontrivial subgroups to which to restrict, and k is strongly flat provided $H^1(G,k) \neq 0$, which is always true.

1.3 A Sufficient Condition for Strong Flatness

Now that we have a condition on modules which implies flatness, we aim to apply this to some representations. We will need the following basic facts from group cohomology:

Theorem 5. *Let $C = \langle \sigma \rangle$ be a cyclic group of order p and X a kC - module, then*

$$H^i(C,X) = \begin{cases} X^C / \mathrm{Tr}_1^C(X) & \text{if } i > 0 \text{ is even}, \\ \ker(\mathrm{Tr}_1^C|_X)/(\sigma - 1)X & \text{if } i \text{ is odd}. \end{cases}$$

Proof. See [10] pg. 6

Let P be a (non-cyclic) p-group and W a right kP-module over a field k of characteristic p. We need to identify non-zero elements in $H^1(P,W)$ which restrict to zero for all maximal subgroups. Let $N \neq M$ denote two maximal subgroups of P. Note that N and M are normal in P. The inflation map gives an injection from $H^1(P/N,W^N)$ to $H^1(P,W)$ (see, for example, [10], Cor. 7.2.3.) Note that P/N is isomorphic to the cyclic group of order p and W^N is a vector space over a field of characteristic p. Thus, unless W^N is a projective P/N-module, $H^1(P/N,W^N)$ is non-zero. Elements in $H^1(P/N,W^N)$ can be represented by vectors in W^N which are in the kernel of the transfer. Assume W^N is not projective and choose $u \in W^N$ so that the equivalence class, $\{u\}$, is non-zero in $H^1(P/N,W^N)$. Clearly the image of $\{u\}$ under the inflation map restricts to zero in $H^1(N,W)$.

Note that $M/(M\cap N)$ is also isomorphic to the cyclic group of order p so that elements of $H^1(M/(M\cap N),W^{M\cap N})$ can be represented by vectors in $W^{M\cap N}$. In [13] Lemma 6.2 it has been shown that the image of $\{u\}$ in $H^1(M,W)$, under the composition of inflation followed by restriction, is zero if and only if u represents zero in $H^1\left(M/(M\cap N),W^{M\cap N}\right)$. This can be used to derive a criterion of strong flatness for non - cyclic p - groups.

Let $g,g'\in P\backslash M$, then $g'=hg^i$ for some $h\in M$ and $1\le i<p$. Since M is normal in P, P acts on W^M and $W^M(g'-1)=W^M(hg^i-1)=$

$$W^M(g^i-1)=W^M(1+g+\cdots+g^{i-1})(g-1)\subseteq W^M(g-1),$$

hence $W^M(g'-1)=W^M(g-1)$ by symmetry. So we can define unambiguously

$$\mathscr{X}_M:=W^M(g-1).$$

Define $X:=N\cap M$, which is a maximal subgroup of M and of N. Let $u,u'\in N\backslash M$, then $N/X=\langle\overline{u}\rangle=\langle\overline{u}'\rangle$ and $u'=xu^i$ for some $x\in X$ and $1\le i<p$. In the same way as above we see $W^X(u'-1)=W^X(u-1)$. So for any maximal subgroup $N\ne M$ we can pick a $u_N\in N\backslash M$ and define

$$\mathscr{Y}_M:=\bigcap_{\substack{N<_{\max}P\\ N\ne M}}W^{N\cap M}(u_N-1).$$

Theorem 6. *For a non-cyclic p - group P the following are equivalent:*

1. $\bigcap_{M<_{\max}P}\ker(\mathrm{res}^P_M|_{H^1(P,W)})\ne 0$
2. *For some* $M\lhd P$ *maximal,* $\mathscr{X}_M<\mathscr{Y}_M\cap W^M$
3. *For all* $M\lhd P$ *maximal,* $\mathscr{X}_M<\mathscr{Y}_M\cap W^M$

Proof. (3) $\Rightarrow$ (2) is clear.

Suppose (2) holds, and let $v\in\mathscr{Y}_M\cap W^M\backslash\mathscr{X}_M$. Pick $g\in P\backslash M$, so that $\overline{g}\in P/M$ generates $P/M\cong C_p$. Since P is non cyclic there exists a maximal subgroup $N\unlhd P$ with $g\in N$ and we define $X:=N\cap M$. Then clearly $N\ne M$ and, by assumption, $v=w'(u_N-1)$ with $w'\in W^X$. As before we see that $W^X(u_N-1)=W^X(g-1)$ so $v=w(g-1)$ with some $w\in W^X$. Hence

$$\mathrm{Tr}_1^{P/M}(v)=v(\overline{g}-1)^{p-1}=w(g-1)^p=w(g^p-1)=0$$

since $g^p\in X$. So $v\in\ker(\mathrm{Tr}_1^{P/M})$. Since v is not in $W^M(g-1)$, and P/M is a cyclic group generated by $\overline{g}$

$$0\ne[v]\in H^1(P/M,W^M).$$

Set $\tau:=\mathrm{inf}^P_M([v])\in H^1(P,W)$. Then $0\ne\tau\in\ker(\mathrm{res}^P_M)$ by [10], Corollary 7.2.3, and by [13], Theorem 6.2, if we can show that

$$0=[v]\in H^1(M'/M\cap M',W^{M\cap M'})$$

for any maximal subgroup $M' < P$, $M \neq M'$, then

$$\tau \in \bigcap_{N<P} \ker(\mathrm{res}^P_N|_{H^1(P,W)}).$$

Let $M' \neq M$ be such a maximal subgroup of P and set $Y := M' \cap M$, then there is $u_{M'} \in M' \backslash M$ and, by assumption $v \in W^Y(u_{M'} - 1)$. Since $\overline{u_{M'}}$ generates M'/Y, we have $0 = [v] \in H^1(M'/Y, W^Y)$. Thus we have proved (2) $\Rightarrow$ (1).

Finally suppose that $0 \neq \tau \in \bigcap_{N<_{\max}P} \ker(\mathrm{res}^P_N|_{H^1(P,W)})$, and let $M \triangleleft P$ be a maximal subgroup. Then by [10], Corollary 7.2.3

$$\tau = \inf^P_M([v])$$

for some $0 \neq [v] \in H^1(P/M, W^M)$. This implies that

$$v \in W^M \backslash W^M(g-1)$$

for each $g \in P \backslash M$. Let $N < P$ be a maximal subgroup with $N \neq M$, set $X := N \cap M$ and let $u_N \in N \backslash M$, then $N/X = \langle \overline{u_N} \rangle$. By assumption $\mathrm{res}^P_N(\tau) = 0$, so by [13], Theorem 6.2,

$$0 = [v] \in H^1(N/X, W^X)$$

i.e. $v \in W^X(u_N - 1)$. So we have shown that for each maximal subgroup M of P we can find an element v with

$$v \in \Big(\bigcap_{M \neq N <_{\max} P} (W^{N \cap M}(u_N - 1)) \Big) \cap W^M \backslash W^M(g-1)$$

i.e. there exists $v \in \mathscr{Y}_M \cap W^M \backslash \mathscr{X}_M$.

1.4 Groups of the Form $C_p \times C_p$

We now apply our results in the simplest case in which the depth of invariant rings is unknown. Let $P \cong C_p \times C_p$ be generated by X and Y, and define maximal subgroups $L := \langle XY \rangle$, $M := \langle X \rangle$, $N := \langle Y \rangle$. Let $W \cong V^*$ be a right P-module over a field k of characteristic p. Every maximal subgroup $H \neq N$ of P is of the form $\langle XY^i \rangle$ for $0 \leq i < p$. Hence, by condition (2) of theorem 6, V is linearly flat if and only if

$$W^N(X-1) < \bigcap_{l=0}^{p-1} W(XY^l - 1) \cap W^N$$

When $p = 2$, the condition takes the particularly simple form

$$W^N(X-1) < W(X-1) \cap W(XY-1) \cap W^N$$

Also by part (3) of theorem 6, we may interchange the roles of maximal subgroups, and when $p = 2$ we will frequently use the alternative condition

$$W^L(X-1) < W(X-1) \cap W(Y-1) \cap W^L$$

There are four infinite families of finite dimensional kP-Modules which are described below. Heller and Reiner [15] have shown that if $p = 2$ then these together with the regular kP-module form a complete set of isomorphism classes of indecomposable kP-modules. In the following, I_n denotes the $n \times n$ - identity matrix and for $\lambda \in k$, J_λ denotes an $n \times n$ matrix in indecomposable (upper triangular) rational canonical form with eigenvalue λ, i.e. an upper triangular matrix with λ on the diagonal, 1 on the super - diagonal and zero elsewhere.

1. For every even dimension $2n$ there are representations $V_{2n,\lambda}$

$$X \sim \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}$$

$$Y \sim \begin{pmatrix} I_n & J_\lambda \\ 0 & I_n \end{pmatrix}.$$

   The dimension of the fixed point space of the corresponding left-module, denoted $\dim(V^P)$ is n.
2. For every even dimension $2n$ there is a representation $V_{2n,\infty}$

$$X \sim \begin{pmatrix} I_n & J_0 \\ 0 & I_n \end{pmatrix}$$

$$Y \sim \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}.$$

   In this case again $\dim(V^P) = n$
3. For every odd dimension $2n+1$ there is a representation $V_{-(2n+1)}$

$$X \sim \begin{pmatrix} I_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_n \\ 0 & & I_{n+1} \end{pmatrix}$$

$$Y \sim \begin{pmatrix} I_n & I_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 & I_{n+1} & \end{pmatrix}$$

   These have $\dim(V^P) = n$

4. For every odd dimension $2n+1$ there is a representation V_{2n+1}

$$X \sim \begin{pmatrix} I_{n+1} & 0...0 \\ & I_n \\ & \\ 0 & I_n \end{pmatrix}$$

$$Y \sim \begin{pmatrix} I_{n+1} & I_n \\ & 0...0 \\ & \\ 0 & I_n \end{pmatrix}$$

For these modules $\dim(V^P) = n+1$.

Theorem 7. *Each even-dimensional module $V_{2n,\lambda}$ with $n \geq 2$ and $\lambda \in k$ or $\lambda = \infty$ is linearly flat. Each odd-dimensional module of the form V_{2n+1} with $n \geq 2$ is linearly flat, while the modules of the form $V_{-(2n+1)}$ are linearly flat if $n \geq p$.*

Corollary 1. *Every non-projective indecomposable modular representation of the group $P := C_2 \times C_2$ is flat.*

Proof. To prove the corollary observe that when $p = 2$, the theorem states that every non-projective indecomposable kP-module of dimension ≥ 4 is linearly flat. But it is also easily seen that each indecomposable kP-module of dimension < 4 in our classification above is trivially flat. Note that there is only one projective indecomposable module, namely the regular module.

Proof. We now show in each of the three cases that there is an element

$$v \in \bigcap_{l=0}^{p-1} W(XY^l - 1) \cap W^N \backslash W^N(X-1).$$

This will prove that the left-module V with $V^* \cong W$ is linearly flat, by theorem 6.

Consider the representation $V := V_{2n,\lambda}$, where for the time being $\lambda \neq \infty$. Then we have

$$W^N = \begin{cases} \langle x_{n+1}, \ldots, x_{2n} \rangle & \lambda \neq 0 \\ \langle x_n, \ldots, x_{2n} \rangle & \lambda = 0 \end{cases}$$

so that

$$W^N(X-1) = \begin{cases} 0 & \lambda \neq 0 \\ \langle x_{2n} \rangle & \lambda = 0 \end{cases}$$

Also for each $0 \leq l \leq p-1$

$$W(XY^l - 1) = \langle (1+l\lambda)x_{n+1} + lx_{n+2}, (1+l\lambda)x_{n+2} + lx_{n+3}, \ldots, (1+l\lambda)x_{2n} \rangle$$

$$W(XY^l - 1) = \begin{cases} \langle x_{n+1}, \ldots, x_{2n} \rangle & \lambda l \neq -1 \\ \langle x_{n+2}, \ldots, x_{2n} \rangle & \lambda l = -1 \end{cases}$$

Now k is a field of characteristic p, so it contains a copy of the field $\mathbb{F}_p$ of order p. If $\lambda \in \mathbb{F}_p \backslash \{0\}$, there is precisely one $l \in \{1, \dots p-1\}$ for which $\lambda l = -1$. If $\lambda = 0$ or if $\lambda \in k \backslash \mathbb{F}_p$, then $\lambda l \neq -1$ for every l. It follows that

$$\bigcap_{l=0}^{p-1} W(XY^l - 1) \cap W^N = \begin{cases} \langle x_{n+1}, \dots, x_{2n} \rangle & \lambda = 0 \text{ or } \lambda \in k \backslash \mathbb{F}_p \\ \langle x_{n+2}, \dots, x_{2n} \rangle & \lambda \in \mathbb{F}_p \backslash \{0\} \end{cases}$$

So we have linear flatness for all λ if $n \geq 2$, and for $n = 1$ when $\lambda \notin \mathbb{F}_p$. $V_{2n,\infty}$ is obtained from $V_{2n,0}$ under the automorphism of P which swaps X and Y, and so the same results hold here.

Now consider the representation $V := V_{2n+1}$. This time, we have

$$W^N = \langle x_{n+1}, \dots, x_{2n+1} \rangle$$

so that $W^N(X-1) = \langle x_{2n+1} \rangle$. Further we have

$$W(XY^l - 1) = \langle l x_{n+2}, x_{n+2} + l x_{n+3}, x_{n+3} + l x_{n+4}, \dots, x_{2n} + l x_{2n+1}, x_{2n+1} \rangle$$

$$W(XY^l - 1) = \langle x_{n+2}, x_{n+3}, \dots, x_{2n+1} \rangle$$

for all $l \in \{0, \dots, p-1\}$. Therefore

$$\bigcap_{l=0}^{p-1} W(XY^l - 1) \cap W^N \backslash W^N(X-1)$$

is non-empty, and hence we have linear flatness, provided that $n \geq 2$.

Finally consider the representation $V := V_{-(2n+1)}$. In this case

$$W^N = \langle x_{n+1}, \dots, x_{2n+1} \rangle$$

so $W^N(X-1) = 0$, and

$$W(XY^l - 1) = \langle l x_{n+1} + x_{n+2}, l x_{n+2} + x_{n+3}, \dots, l x_{2n} + x_{2n+1} \rangle$$

Let us denote each space $W(XY^l - 1)$ by the shorthand W_l. Since $W_l \subset W^N$ for each $l = 1, \dots, p-1$, it is enough to show that $\bigcap_{l=0}^{p-1} W_l \neq 0$. It is easy to see that each W_l is a hyperplane in the $n+1$-dimensional space W^N. It then follows that

$$\dim \bigcap_{l=0}^{p-1} W_l \geq n + 1 - p$$

And so $V_{-(2n+1)}$ is linearly flat when $n \geq p$.

1.5 Decomposable Representations

We have proved that all non-projective indecomposable representations of $C_2 \times C_2$ are flat, and that provided $\dim(V^P) + 2 < \dim(V)$, any representation containing the indecomposable V as a direct summand will also be flat. We now go on to classify the modular representations of $C_2 \times C_2$ which are strongly flat, by investigating which direct summands may appear in their symmetric algebras. Before we do so, we restate and extend theorem 9 specifically for the case $p = 2$.

Theorem 8. *Let V be an indecomposable representation of $P := C_2 \times C_2$ over a field of characteristic 2. Then:*

1. *If $V \cong V_{2,\lambda}$ for $(\lambda \neq 0, 1, \infty)$, $V_{4,\lambda}$, or V indecomposable of dimension ≥ 5, then V is linearly flat.*
2. *If $\dim(V) \leq 4$ and V is not projective, then V is trivially flat*
3. *If $V \cong$ the projective indecomposable of dimension 4, then V is not flat.*
4. *The representations $V_{2,0}, V_{2,1}, V_{2,\infty}, V_3$ and V_{-3}, though flat, are not strongly so.*

Note that since this is a p-group in characteristic p, there is only one projective indecomposable, namely the regular representation. From now on this is abbreviated to $\overline{V}_4$

Proof. : (1) and (2) are clear. The remaining statements will be proven in the following lemmas.

$\overline{V}_4$ is equivalent to a permutation representation. We can think of X and Y acting on V^* as the permutations $(12)(34)$ and $(13)(24)$ respectively, by which we mean that

$$x_1X = x_2, x_2X = x_1, x_3X = x_4, x_4X = x_3$$

and the action of Y calculated similarly. It is clear that $\dim(V^P) = 1$, therefore, V is either flat or R^P is Cohen-Macaulay. In this case we can find explicit generators for R^P in order to show that the latter holds, and V is not flat.

Remark: The ring of invariants for $C_2 \times C_2$ and $\overline{V}_4$ has been determined in [1], where k is assumed to be the field of order 2. The version we present here for convenience an completeness, assumes k to be any field of characteristic 2. We use the following Lemma:

Lemma 2. *Let $R := S(V^*) = k[x_1, x_2, x_3, x_4]$ with $V \cong \overline{V}_4$. Then the ring of invariants R^M may be described as*

$$S := k[a, b, c, d](1, \beta_1)$$

where $a := x_1 + x_2, b := x_3 + x_4, c := x_1x_2, d := x_3x_4$ form an hsop and $\beta_1 := x_1x_3 + x_2x_4$, with $\beta_1^2 = \beta_1 ab + a^2d + b^2c$.

Proof. This is a special case of [19], Proposition 11.

To find R^P, we apply the lemma twice and obtain

Lemma 3. *We keep the notation from 2. Then the ring of invariants R^P is Cohen - Macaulay and can be described as*

$$R^P = k[a+b, c+d, ab, cd](1, \beta_1)(1, ac+bd)$$

$$= k[x_1+x_2+x_3+x_4, x_1x_2+x_3x_4, (x_1+x_2)(x_3+x_4), x_1x_2x_3x_4](1, \beta_1, \beta_2, \beta_1\beta_2)$$

where $\beta_2 = x_1^2x_2 + x_1x_2^2 + x_3^2x_4 + x_3x_4^2$. In particular R^P is not flat.

Proof. The quotient P/M acts on $R^M = k[a,b,c,d](1,\beta_1)$ by swapping a with b, swapping c with d, and by fixing β_1. Hence 2 gives the first equality, with primary invariants of degrees 1,2,2 and 4. Since the product of these degrees is 16 and the minimal set of secondary invariants has $4 = 16/|P|$ elements, we conclude by [6], Theorem 3.7.1, that R^P is a Cohen-Macaulay ring. Since $\mathrm{Dim}(R^P) = 4$ and $\dim(\overline{V}_4^P) = 1$, the ring R^P cannot be flat.

The proof of (4) will again be broken down into two lemmas. Recall that in order to prove strong flatness we would need to find an element of

$$R(X+1) \cap R(Y+1) \cap R^L \backslash R^L(X+1).$$

If $V \cong V_{2,0}$ then $R(Y+1) = 0$, if $V \cong V_{2,\infty}$ then $R(X+1) = 0$ and if $V \cong V_{2,1}$ then $R^L(X+1) = R(X+1)$. So none of these modules are strongly flat as P-modules.

Lemma 4. *Let $R := S(V^*) = k[x_1,x_2,x_3]$ with $V \cong V_3$. Then the ring of invariants R^P is polynomial of the form $B := k[u,v,w]$ $u := x_3$, $v := x_1(x_1+x_3)$ and $w := x_2(x_2+x_3)$. The representation V_3 is flat but not strongly flat.*

Proof. Note that P acts as follows

$$X: \ x_1 \mapsto x_1, \ x_2 \mapsto x_2 + x_3, \ x_3 \mapsto x_3$$

$$Y: \ x_1 \mapsto x_1 + x_3, \ x_2 \mapsto x_2, \ x_3 \mapsto x_3$$

$$XY: \ x_1 \mapsto x_1 + x_3, \ x_2 \mapsto x_2 + x_3, \ x_3 \mapsto x_3$$

hence $u,v,w \in R^P$ and R is integral over B (e.g. $x_1^2 + x_1u - v = 0$); moreover the product of their degrees is $4 = |P|$, hence by [6], Theorem 3.7.5, $R^P = k[u,v,w]$ as required. Since $R = B(1,x_1,x_2,x_1x_2)$ as a B - module and $[\mathrm{Frac}(R) : \mathrm{Frac}(B)] = 4$, we see that $R = B \oplus Bx_1 \oplus Bx_2 \oplus Bx_1x_2$. Using this decomposition it is easily seen that

$$R^M = B(1,x_1), \ R^N = B(1,x_2) \text{ and } R^L = B(1,x_1+x_2).$$

Now we obtain immediately

$$R^L(X+1) = Bx_3$$

$$R(X+1) = R^M(x_3) = B(x_3, x_1x_3)$$

$$R(Y+1) = R^N(x_3) = B(x_3, x_2x_3).$$

Therefore $R(X+1) \cap R(Y+1) = Bx_3 = R^L(X+1)$, and so there cannot be an element in $R(X+1) \cap R(Y+1) \cap R^L \backslash R^L(X+1)$.

Lemma 5. *Let $R := S(V^*) = k[x_1,x_2,x_3]$ with $V \cong V_{-3}$. Then the ring of invariants R^P is polynomial of the form $B := k[x_2,x_3,z]$ with $z := N_P(x_1) := \prod_{g\in P}(x_1 g) = x_1^4 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2$. The representation V_{-3} is flat but not strongly flat.*

Proof. Clearly $B \subseteq R^P$ and the equation

$$x_1^4 = x_1^2(x_2^2 + x_3^2 + x_2x_3) + x_1(x_2^2x_3 + x_2x_3^2) + z$$

shows that R is integral over B. Since the degree product of x_2, x_3 and z is $4 = |P|$, again by [6], Theorem 3.7.5, we have $B = R^P$ with a decomposition

$$R = B[x_1] = B \oplus Bx_1 \oplus Bx_1^2 \oplus Bx_1^3.$$

Let $u := N_M(x_1) := x_1(x_1+x_3)$, $v := N_N(x_1) := x_1(x_1+x_2)$ and $w := N_L(x_1) := x_1(x_1+x_2+x_3)$. Since $z = N_P(x_1) = N_N(u) = u \cdot (u)Y = u(u + x_2^2 + x_2x_3)$, it follows that $u^2 \in B + Bu = B \oplus Bu$. Similarly $v^2 \in B \oplus Bv$ and $w^2 \in B \oplus Bw$ and we find

$$R^M = B[u] = B \oplus Bu$$

$$R^N = B[v] = B \oplus Bv$$

$$R^L = B[w] = B \oplus Bw.$$

From this we see

$$R(X+1) = R^M(x_3) = B(x_3, ux_3)$$

$$R(Y+1) = R^N(x_2) = B(x_2, vx_2)$$

$$R(XY+1) = R^L(x_1) = B(x_1, wx_1)$$

$$R^L(X+1) = B((w)(X+1)) = B(x_2x_3).$$

Suppose $s \in R(X+1) \cap R(Y+1) \subseteq B$, then $s = x_3 r = x_2 r'$ with $r \in R^M$ and $r' \in R^N$, hence $r'/x_3 \in R$, so $s = x_2x_3s'$ with P - fixed $s' \in R$ and we conclude $s' \in B$ and $s \in Bx_2x_3 = R^L(X+1)$. This shows that V_{-3} cannot be strongly flat.

This last lemma also terminates the proof of 8.

Corollary 2. *A complete list of indecomposable kP-modules for $P := C_2 \times C_2$ whose polynomial invariants are Cohen-Macaulay is given by*

$$V_1, V_{2,\lambda}, V_{-3}, V_3, V_{4,\lambda}, \overline{V}_4, V_5$$

Proof. We have stated already that the modules $V_1, V_{2,\lambda}, V_{-3}, V_3$ are trivially flat. Observe that $\dim(V_{4,\lambda}^P) = 2$ and $\dim(V_5^P) = 3$, so both of these are trivially flat.

That $\overline{V}_4$ gives rise to a Cohen-Macaulay invariant ring is proved in Theorem 8. Finally we check the statement of Theorem 1, to see that every other indecomposable representation is flat, but not trivially so.

Definition 2. We denote by NSF the following set of isomorphism types of $k(C_2 \times C_2)$ - modules:

$$\{V_1, V_{2,1}, V_{2,0}, V_{2,\infty}, V_3, V_{-3}, \overline{V}_4\}.$$

Corollary 3. *The symmetric algebras of the representations V_3,V_{-3} and $\overline{V}_4$ contain only indecomposable direct summands of the form $V_1, V_{2,1}, V_{2,0}, V_{2,\infty}, V_3$,$V_{-3}$ and $\overline{V}_4$.*[1]

Proof. Let $V \in NSF$. All indecomposable modules not in NSF are linearly flat. If $S(V^*)$ contains a direct summand W^* for which W is linearly flat, then V is strongly flat. Observing that NSF is closed under the operation of taking duals (where $V_3^* \cong V_{-3}$, and the rest are self-dual) completes the proof.

Now if we can determine precisely which indecomposable summands appear in the symmetric algebras of modules in NSF, and how the tensor products decompose into indecomposable summands, we will then be able to list all possible kP-modules which are strongly flat.

Theorem 9. *Let* $\mathrm{add}(NSF)$ *denote the set of modules which are direct sums of modules in NSF. Let W be any kP-module. Then W is strongly flat unless $W \in \mathrm{add}(NSF)$ and neither $V_3 \oplus V_3$ nor $V_{-3} \oplus V_{-3}$ is a direct summand of W.*

Proof. If $W \notin \mathrm{add}(NSF)$, then W must contain a direct summand which is strongly flat. Consequently, W is strongly flat. So we assume $W \in \mathrm{add}(NSF)$. We write

$$W = aV_1 \oplus b_0V_{2,0} \oplus b_1V_{2,1} \oplus b_\infty V_{2,\infty} \oplus c_+V_3 \oplus c_-V_{-3} \oplus d\overline{V}_4$$

So that

$$W^* \cong aV_1 \oplus b_0V_{2,0} \oplus b_1V_{2,1} \oplus b_\infty V_{2,\infty} \oplus c_+V_{-3} \oplus c_-V_3 \oplus d\overline{V}_4$$

and $S(W^*) \cong$

$$S(V_1)^{\otimes a} \otimes S(V_{2,0})^{\otimes b_0} \otimes S(V_{2,1})^{\otimes b_1} \otimes S(V_{2,\infty})^{b_\infty} \otimes S(V_{-3})^{\otimes c_+} \otimes S(V_3)^{\otimes c_-} \otimes S(\overline{V}_4)^{\otimes d}$$

In order to determine which isomorphism classes of modules may appear as summands in the above expression, we need to know which summands appear in the symmetric algebras of the modules in NSF, and also how their tensor products decompose. Clearly the symmetric algebras of the non-faithful modules $V_{2,0}, V_{2,1}$ and $V_{2,\infty}$ contain as direct summands only themselves and the trivial module, as any direct summands in the symmetric algebra must also be non-faithful. Still more obviously, the symmetric algebra of V_1 contains only copies of the trivial module. In

[1] V_1 denotes the trivial module

order to ascertain which modules appear in the symmetric algebras of V_3, V_{-3} and $\overline{V}_4$, we use [16], Theorem 1.2. This tells us that all summands which do appear as summands of the symmetric algebra can be found in degrees $\leq 2^n - n - 1$, where n is the dimension of the representation. This allows for computation, using the meataxe in MAGMA [4]. The result is:

- $S(V_{-3})$ contains summands isomorphic to $V_1, V_{2,1}, V_{2,\infty}, V_{2,0}, V_{-3}$ and $\overline{V}_4$
- $S(V_3)$ contains summands isomorphic to V_1, V_3 and $\overline{V}_4$
- $S(\overline{V}_4)$ contains summands isomorphic to $V_1, V_{2,1}, V_{2,\infty}, V_{2,0}$ and $\overline{V}_4$

The first two results require a decomposition of the first four symmetric powers, while the third requires a decomposition of the first 11. This takes considerable computation time. However there is a far easier way to obtain this result - $\overline{V}_4$ is a permutation module of dimension 4, so each indecomposable summand in its symmetric algebra has dimension dividing 4. This tells us that $S(\overline{V}_4)$ cannot contain summands isomorphic to V_3 or V_{-3} and the result now follows from Corollary 3. The following table lists the decompositions of tensor products in NSF. We omit from the table results concerning V_1, as $W \otimes V_1 = W$ for any kP-module W.

Table 1.1 Decomposing Tensor Products of kP-modules

$\otimes$	$V_{2,1}$	$V_{2,\infty}$	$V_{2,0}$	V_3	V_{-3}	$\overline{V}_4$
$V_{2,1}$	$V_{2,1}^{\oplus 2}$	$\overline{V}_4$	$\overline{V}_4$	$V_{2,1} \oplus \overline{V}_4$	$V_{2,1} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
$V_{2,\infty}$		$V_{2,\infty}^{\oplus 2}$	$\overline{V}_4$	$V_{2,\infty} \oplus \overline{V}_4$	$V_{2,\infty} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
$V_{2,0}$			$V_{2,0}^{\oplus 2}$	$V_{2,0} \oplus \overline{V}_4$	$V_{2,0} \oplus \overline{V}_4$	$\overline{V}_4^{\oplus 2}$
V_3				$\overline{V}_4 \oplus V_5$	$V_1 \oplus \overline{V}_4^{\oplus 2}$	$\overline{V}_4^{\oplus 3}$
V_{-3}					$\overline{V}_4 \oplus V_{-5}$	$\overline{V}_4^{\oplus 3}$
$\overline{V}_4$						$\overline{V}_4^{\oplus 4}$

Note that the last column of that list is explained by the fact that the tensor product of any module with a projective one is projective and $\overline{V}_4$ is the only indecomposable projective module for $k(C_2 \times C_2)$. We conclude that the modules in the set $\mathrm{add}\{V_1, V_{2,1}, V_{2,\infty}, V_{2,0}, \overline{V}_4\}$ form a closed system under taking symmetric algebras. Moreover all the modules in the decomposition of $S(W^*)$ belong to NSF unless either c_+ or c_- is at least two. If $c_+ \geq 2$, then $S(W^*)$ contains a direct summand isomorphic to $V_{-5} \cong V_5^*$ and W is therefore strongly flat. Similarly if $c_- \geq 2$, then $S(W^*)$ contains a direct summand isomorphic to $V_5 \cong V_{-5}^*$ and again W is strongly flat.

It must be stressed that in this paper we have determined the depth of R^G for any indecomposable representation of $P := C_2 \times C_2$, (and so classified its flat indecomposable representations), and also classified all the strongly flat representations of this group. We have not, however, classified all the flat representations of this group.

Consider, for example, the kP-module $W := V_3 \oplus V_{2,1} \oplus V_{2,1}$. Then $W \in \text{add}(NSF)$ and neither $V_3 \oplus V_3$ nor $V_{-3} \oplus V_{-3}$ is a direct summand of W, so by Theorem 9, W is not strongly flat. A calculation in MAGMA establishes that the depth of the corresponding invariant ring R^G is 6, and so W is in fact flat. We may easily verify that W is not trivially flat. So then, it must be the case (by Theorem 3) that there is a cohomology class $0 \neq \alpha \in H^1(P,R)$ which is annihilated by each element of the relative transfer ideal. At the same time, the intersection of the kernels of the restrictions to each maximal subgroup is zero. Finding such an α directly is, at least for the moment, beyond the scope of the methods presented in this paper.

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